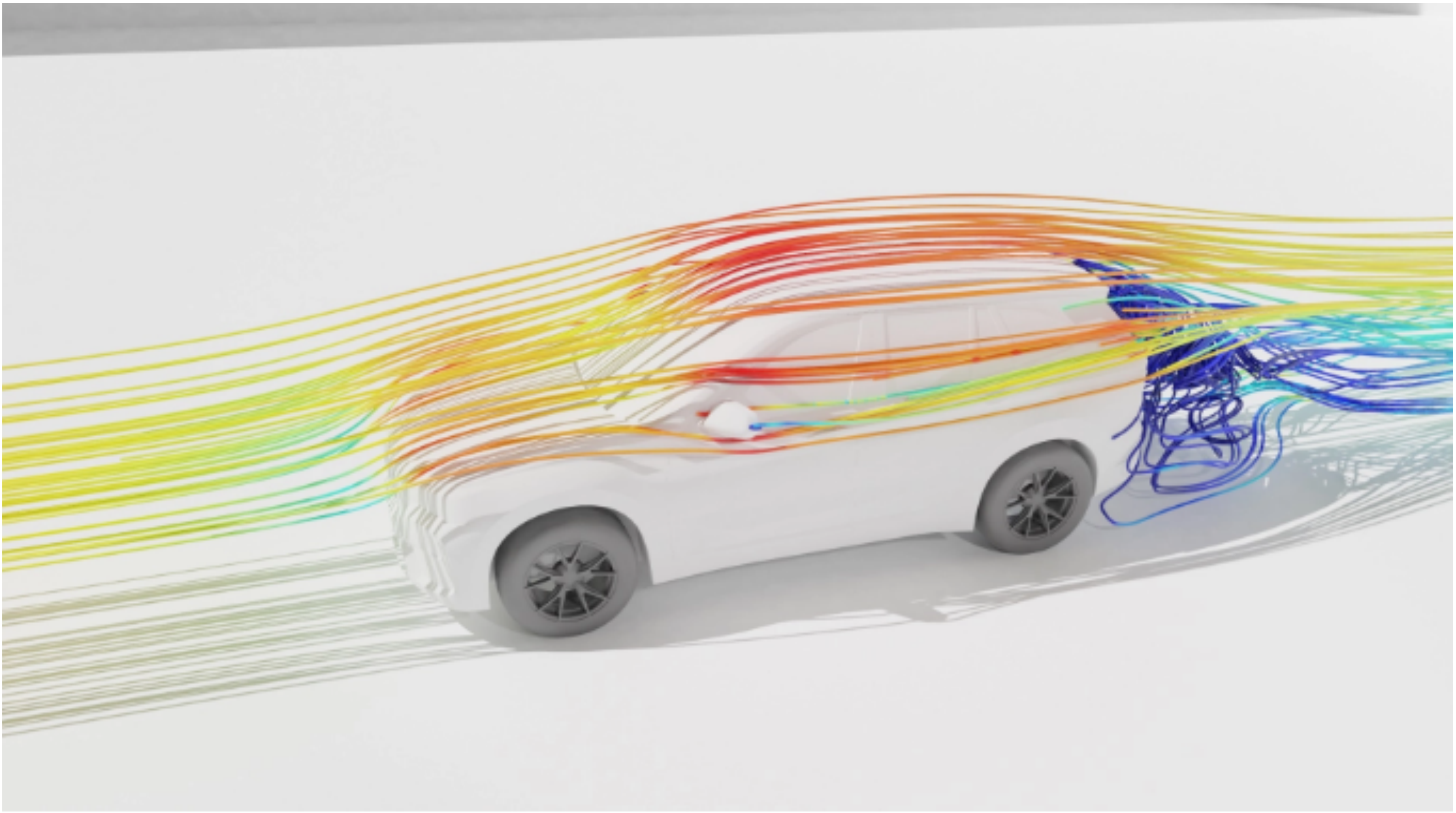
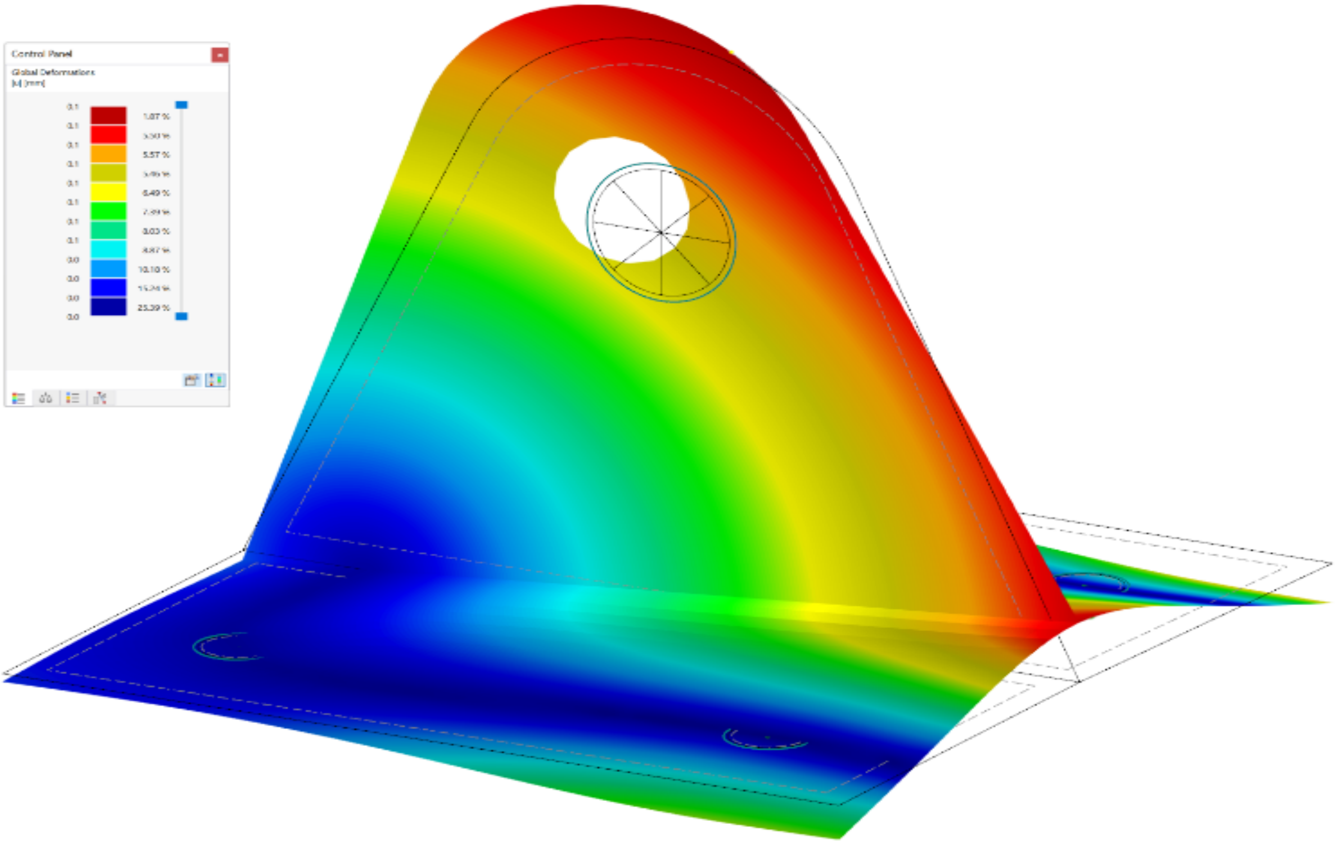


Continuum Mechanics

Fundamental Concepts

Mr. Karim Zennadi



Preface

Continuum mechanics studies the deformation of solid and fluid bodies over length scales ranging from a few μm to several km. The main working assumption is contained in the title of this book. It is assumed that the density of atoms is sufficiently high so that the interactions between individual atoms or molecules do not need to be considered explicitly.

Continuum mechanics is based on an axiomatic formulation of the fundamental laws of mechanics (conservation of mass, conservation of momentum, and power). From this axiomatic formulation, a rigorous mathematical framework leads to a precise description of deformations, stresses (internal forces within a solid), and to the formulation of constitutive laws describing the behavior of solids and fluids.

In this course book, we will limit ourselves to the reversible elastic behavior of solids. This behavior adequately describes the use of structures in most everyday engineering applications. However, the production of materials and the forming of structures often involve plastic, elastoplastic, and elastoviscoplastic behaviors.

The course on continuum mechanics relies on a rather demanding mathematical formalism. Nevertheless, applications of continuum mechanics are part of the daily practice of every mechanical engineer.

At the end of each chapter, solved exercises of varying length are provided in order to review many of the concepts presented in the course.

This course manuscript was prepared primarily based on the following references, which provide modern, concise, and rigorous presentations of continuum mechanics:

- Erick Ringot, *Continuum Mechanics Notes: Linear Elasticity*, Paul Sabatier University, Toulouse 3, 2010.
- Frédéric Golay and Stéphane Bonelli, *Continuum Mechanics*, ISITV, 2011.

- Ziad Moumni, *Continuum Mechanics*, ENSTA, 2005.
- Jean Salençon, *Continuum Mechanics, Volume 1: General Concepts*, École Polytechnique, September 2007.

Writing lecture notes on continuum mechanics for a core engineering curriculum is not an easy task. In addition to my experience as a teacher, I have been greatly assisted in this endeavor by several colleagues who kindly took the time to provide their comments and suggestions on this document.

There are certainly still many imperfections in this text, and I would be grateful if readers would kindly point them out.

M. Karim Zennadi, Fevrier 2026.
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Dedication

Through this work, I wish to pay a profound tribute to my brother,

Raouf Zennadi

who passed away in January 2024.

My dear brother,

I still carry the overwhelming emotion of your passing.
It left behind an immense void,
yet brought the painful relief of knowing you are finally at peace.
It is hard to admit, but it is the truth:
watching you suffer had become unbearable.
I wanted to hold onto you, even though, deep down, I knew the outcome.

You departed far too soon, leaving a deafening silence,
But the radiance of your smile remains our most precious gift.

M. Karim Zennadi, Novembre 2024.
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Contents

I INTRODUCTION A LA MMC

1 Elements of Tensor Calculus 19
1.1 Introduction to Tensors 19
1.1.1 Introduction 19
1.1.2 Order of a Tensor 19
1.2 Vectors and Tensors 20
1.2.1 Vector 20
1.2.2 Linear Mapping from ξ to ξ 21
1.2.3 Bilinear Forms 21
1.2.4 Tensors 22
1.2.5 Change of Reference Frame 22
1.3 Permutations and Determinants 23
1.3.1 Permutation Symbols (Levi-Civita Symbol) 23
1.3.2 Determinant of a Matrix 24
1.3.3 Adjoint of an Antisymmetric Tensor 24
1.4 Vector Calculus and Vector Analysis 24
1.4.1 Vector Calculus 24
1.4.2 Vector Analysis 25
1.4.3 Integral Transformation 25

2 Introduction to Continuum Mechanics 27
2.1 Some Definitions 27
2.2 The Place of Continuum Mechanics within Mechanics 27
2.3 Main Assumptions of Continuum Mechanics 28

2.4 Typology of Continuous Media 30
2.5 Applications of Continuum Mechanics 30
2.6 Spirit of the Course 31

II CINÉMATIQUE

3 The Continuum Medium: A Modeling Framework 35
3.1 Scale, Modeling, Validation 35
3.2 Concepts and Their Formulation 36
3.2.1 The Guiding Idea 36
3.2.2 Configurations of the System 36
3.3 Lagrangian and Eulerian Points of View 37
3.4 Lagrangian Description 37
3.4.1 Definition 37
3.4.2 Continuity Assumptions 38
3.4.3 Relevance of the Model: Weakening of the Continuity Assumptions 39
3.4.4 Physical Interpretation of the Lagrangian Description: Trajectories 40
3.4.5 Streaklines 41
3.4.6 Velocity of a Particle 41
3.5 Eulerian Description 41
3.5.1 Definition 41
3.5.2 Determination of Trajectories 43
3.5.3 Streamlines 44
3.5.4 Steady Motions 45
3.6 Exercise 46
3.6.1 Statement 46
3.6.2 Solution 46

4 Study of the Deformation of a Continuum Medium 51
4.1 Introduction 51
4.2 Gradient of a Transformation 52
4.2.1 Transformation Gradient Tensor 52
4.2.2 Convective Transport of a Vector 56
4.3 Stretch Tensor 56
4.3.1 Stretch Tensor 57
4.3.2 Stretch in a Given Direction 59
4.3.3 Shear Between Two Orthogonal Directions 60
4.3.4 Evolution of Volumes 61
4.3.5 Polar Decomposition 62
4.4 Strain Tensors 63
4.4.1 Green-Lagrange Tensor 63
4.5 Formulation in Terms of Displacements 66
4.5.1 Displacements 66
4.5.2 The Transformation Gradient Tensor 66
4.5.3 The Cauchy-Green Stretch Tensor 67
4.5.4 The Green-Lagrange Strain Tensor 67

4.6 Small-Strain Assumption 67

4.6.1 Consequences . . . 67

4.7 Small-Transformation Assumption 67

4.7.1 Consequences . . . 68

4.8 Compatibility Conditions 68

4.8.1 Uniqueness of the Displacement Field . . . 68

4.8.2 Existence of a Displacement Field: Compatibility Conditions . . . 69

4.9 Exercise 70

4.9.1 Statement . . . 70

4.9.2 Solution . . . 71

5 Kinematics of the Continuum Medium . . . 75

5.1 Introduction 75

5.2 Velocity Gradient Tensor 75

5.3 Strain-Rate Tensor 77

5.3.1 Rate of Change of Lengths . . . 77

5.3.2 Rate of Shear of Two Directions . . . 78

5.3.3 Rate of Change of Volume . . . 79

5.4 Rotation-Rate Tensor 79

5.5 Material Derivative 80

5.5.1 Intuitive Notion . . . 80

5.5.2 Material Derivative of a Point Function . . . 80

5.5.3 Material Derivative of a Volume Integral . . . 81

5.6 Exercise 85

5.6.1 Statement . . . 85

5.6.2 Solution . . . 85

III EFFORTS DANS LES MILIEUX CONTINUS

6 Conservation of Mass: Continuity Equation . . . 91

6.1 Introduction 91

6.2 First Case 92

6.3 Second Case 93

7 Stress . . . 95

7.1 Objective 95

7.2 Interactions – Newtonian Framework 95

7.2.1 Space . . . 95

7.2.2 Time . . . 95

7.2.3 Origin of Forces . . . 96

7.2.4 Fundamental Principle . . . 96

7.2.5 Action–Reaction . . . 97

7.3 **Stress Field** **98**
7.3.1 Partitioning of a Solid 98
7.3.2 Hypothesis on the Nature of Internal Forces 98
7.3.3 Normal Stress and Shear Stress 100
7.4 **Properties of Stress** **100**
7.4.1 Stress Tensor 100
7.4.2 Dynamic Equilibrium of a Solid Domain 101
7.5 **Diagonalization of the Stress Tensor** **104**
7.5.1 Principal Stresses 104
7.5.2 Principal Stress Directions 104
7.5.3 Invariants of the Stress Tensor 105
7.5.4 Spherical and Deviatoric Parts of a Stress Tensor 105
7.6 **Graphical Representation of a Stress State** **107**
7.6.1 MOHR's Three-Circle Diagram 107
7.6.2 MOHR's Circle 108
7.7 **Boundary Conditions of the Solid Expressed in Terms of Stress** **112**
7.8 **Exercise** **112**
7.8.1 Statement 112
7.8.2 Solution 113

IV ÉLASTICITÉ

8 **Linear Elastic Constitutive Law 121**
8.1 **Introduction** **121**
8.2 **Hypotheses** **123**
8.2.1 Experimentation 123
8.2.2 Behavior of steel (tensile test) 123
8.3 **Formulations of the linear elastic law** **124**
8.3.1 HOOKE–DUHAMEL law 124
8.3.2 GABRIEL LAMÉ law 126
8.3.3 Relations between coefficients 126
8.3.4 Cross formulations 127
8.4 **Power, work, energy** **127**
8.4.1 Principle of virtual powers (PVP) 127
8.4.2 Elastic strain energy density 129
8.4.3 Theorem of mechanical energy 130

9 **Methods for Solving Elasticity Problems 131**
9.1 **Elasticity Problem** **131**
9.1.1 Problem statement 131
9.1.2 Objective 132
9.1.3 Summary of the available equations 132
9.2 **NAVIER Method** **134**
9.2.1 Principle 134
9.2.2 Equations 134
9.2.3 Kinematically admissible displacement field 135

9.3 BELTRAMI–MITCHEL Method 135
9.3.1 Principle 135
9.3.2 Equations 135
9.3.3 Case of uniform body forces 136
9.3.4 Statically admissible stress field 137
9.4 Uniqueness of the Solution of an Elasticity Problem 137
9.4.1 Principle of superposition 137
9.4.2 Uniqueness of the solution 137
9.4.3 Associated KA and SA fields 137
9.5 RITZ Method 137
9.5.1 Theorem 138
9.5.2 Application 139
9.5.3 Finite element method 140
9.6 Two-Dimensional Elasticity 141
9.6.1 Form of the tensors – reduced tensors 142
9.6.2 Compatibility equation 143
9.6.3 Trace of the tensors 144
9.6.4 Equilibrium 144
9.6.5 Stress–strain relation 144
9.6.6 Volumetric energy 145
9.6.7 BELTRAMI equations 145
9.6.8 AIRY function 146
9.7 Axisymmetric Elasticity 148
9.8 Exercise 148
9.8.1 Statement 148
9.8.2 Solution 149

Bibliography 153

List of Figures

3.1 Cold or hot forming . . . 35
3.2 Configurations of the system. . . . 37
3.3 Asymmetric punching of a plasticine block . . . 39
3.4 Flow around a flat plate. . . . 40
3.5 Trajectory of a particle. . . . 40
3.6 Streakline (dashed) . . . 41
3.7 Visualization of streaklines originating from the leading edge of a delta wing using dye injection. (ONERA image.) . . . 42
3.8 Eulerian description. . . . 43
3.9 Streamlines at time T (dashed lines: trajectories of two particles) . . . 44
3.10 Laser-sheet visualization of the streamlines of the transverse flow around the Concorde. (ONERA image.) . . . 45
3.11 Flow around a high-speed train nose model. (ONERA image.) . . . 45
3.12 Trajectories of the three points A, B, and C for $t = 0..5s$ . . . 48
3.13 Streakline associated with the origin at time $T = 3$ . . . 48
3.14 Streamlines at time $\tau = 3$ . . . 49

4.1 Grid of lines drawn on a solid before indentation and after indentation by a cylinder. . . . 51
4.2 Initial and current configurations of a deformed solid. . . . 52
4.3 Example of a particular transformation: simple shear with translation. . . . 54
4.4 Indentation of a solid by an infinite cylinder, with the undeformed configuration taken as reference . . . 55
4.5 Indentation of a solid by an infinite cylinder, with the deformed configuration shown . . . 55
4.6 Transition from the representation of finite quantities to differential quantities. . . . 56
4.7 Transport of an elementary vector. . . . 57
4.8 Indentation of a solid by an infinite cylinder. Reference configuration and deformed configuration. . . . 57
4.9 Evolution of lengths. . . . 59
4.10 Evolution of angles. . . . 60
4.11 Evolution of volumes. . . . 61

4.12 Transformation of the principal basis of the Cauchy stretch tensor. 63
4.13 Transformation of the principal basis of the Cauchy stretch tensor. 64
4.14 Displacement of a material point. 66
4.15 Variation of the volume $\gamma(t)$ as a function of time. 71

5.1 Kinematics of the continuum medium. 76
5.2 Definition of the velocity gradient tensor. 77
5.3 Rate of shear. 78
5.4 Material derivative of a volume integral. 82
5.5 Presence of a discontinuity. 84

6.1 Mass of a continuum medium. 91
6.2 Presence of a discontinuity. 93

7.1 Fundamental theorem. 96
7.2 Solid considered as the union of its two parts. 97
7.3 Solid decomposed into two parts. 98
7.4 Solid subjected to forces. 99
7.5 Section of the solid. 99
7.6 Tetrahedral volume element. 101
7.7 Subdomain extracted from a solid. 102
7.8 Reciprocity of shear stresses. 104
7.9 Principal stress and principal stress direction. 105
7.10 Normal stress and shear stress. 107
7.11 MOHR's three-circle diagram (for stresses). 108
7.12 Plane stress state (case of the plane perpendicular to axis III). 109
7.13 MOHR's circle for stresses. 110
7.14 Normal and tangent vectors. 111
7.15 Construction of MOHR's circle when the principal stresses are unknown. 112
7.16 Construction of MOHR's circle. 115

8.1 Tensile curve. 121
8.2 Strain–number of cycles curve. 123

9.1 Data of an elasticity problem. 131
9.2 Methodology for solving an elasticity problem. 134
9.3 Work versus potential of a force – spring case. 139
9.4 Mesh of a shell. 141
9.5 Plane strain and plane stress states. 142
9.6 Example of an axisymmetric problem. 148
9.7 Elasticity problem. 149
9.8 Section of the block. 149

Preamble

Skills Acquisition

Continuum mechanics, and linear elasticity in particular, forms the foundation of many mathematical theories that support structural design in mechanical engineering and civil engineering. The aim of this course is to enable students to acquire new skills that will allow them to understand the terminology, concepts, and methods used in computational codes and international design standards. These fundamental tools, which every competent engineer must master, are thus made accessible to mechanical engineering students.

Content

The objective of **Continuum Mechanics (CM)** is the study of the behavior of continuous media under the assumption of continuity. This first-year Mechanical Engineering course is intentionally limited to the study of elastic solids under the assumption of small perturbations.

Overall, the course covers the following topics:

1. Kinematics explains the hypothesis of continuity and deals with the displacements and deformations of solids;
2. Statics addresses the question of internal forces and introduces the concept of stresses and their properties;
3. Material constitutive laws establish the relationship between stresses and strains. The elastic law of homogeneous isotropic media is presented in detail;
4. The formulation of continuum mechanics problems and the solution methods used in elasticity.

Prerequisites

Mathematics

Vector and affine spaces. Matrix algebra: matrices, characteristic polynomial, eigenvalues and eigenvectors. Elementary Euclidean geometry: right triangle, circle, trigonometry. Cross product, dot product. Equation and parametric representation of a curve and a surface. Cartesian and cylindrical coordinates. Functions and derivatives. Functions of several variables, partial derivatives, differentials. Vector functions and vector analysis.

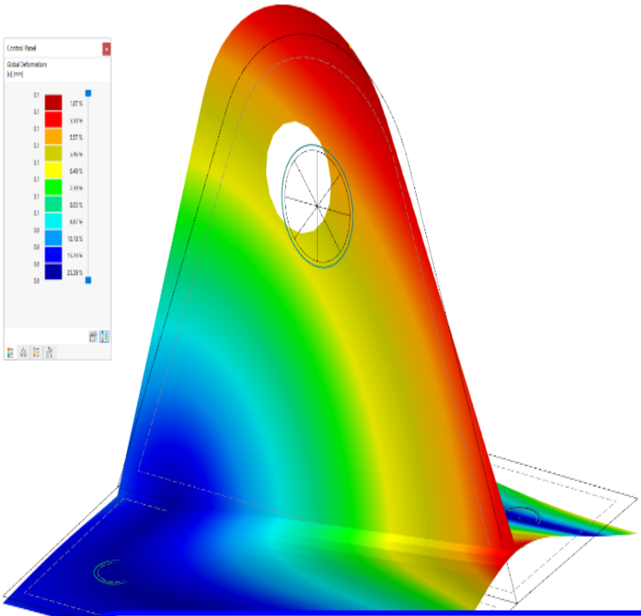
Mechanics

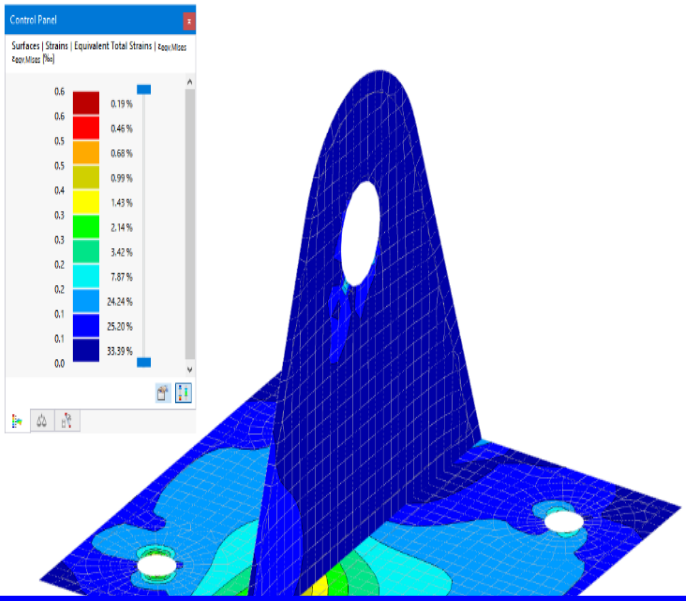
Force, moment of a force, wrench. Mechanics of rigid bodies: kinematic wrench, dynamic wrench. Fundamental principle of dynamics. Statics.

INTRODUCTION A LA MMC

1 Elements of Tensor Calculus 19
1.1 Introduction to Tensors
1.2 Vectors and Tensors
1.3 Permutations and Determinants
1.4 Vector Calculus and Vector Analysis

2 Introduction to Continuum Mechanics . 27
2.1 Some Definitions
2.2 The Place of Continuum Mechanics within Mechanics
2.3 Main Assumptions of Continuum Mechanics
2.4 Typology of Continuous Media
2.5 Applications of Continuum Mechanics
2.6 Spirit of the Course





1. Elements of Tensor Calculus

Continuum mechanics makes extensive use of scalar, vector, and tensor fields. These essential mathematical tools not only make it possible to establish fundamental results independently of the chosen reference frame, but they also give the formulas that express them a remarkable conciseness. As a result, attention can be focused on the physical phenomena they represent rather than on the equations themselves.

Scalars, vectors, and tensors indeed possess the property of being invariant under a change of basis. Thanks to these quantities, the equations of mechanics can be written in an intrinsic form, that is, independently of the chosen basis.

In this course, we will not make use of the most general form of tensor calculus. We will only use orthogonal coordinate systems, possibly curvilinear (for example cylindrical or spherical coordinate systems), which allows considerable simplifications without introducing overly restrictive assumptions. Moreover, all vectors and tensors considered will have real components.

1.1 Introduction to Tensors

1.1.1 Introduction

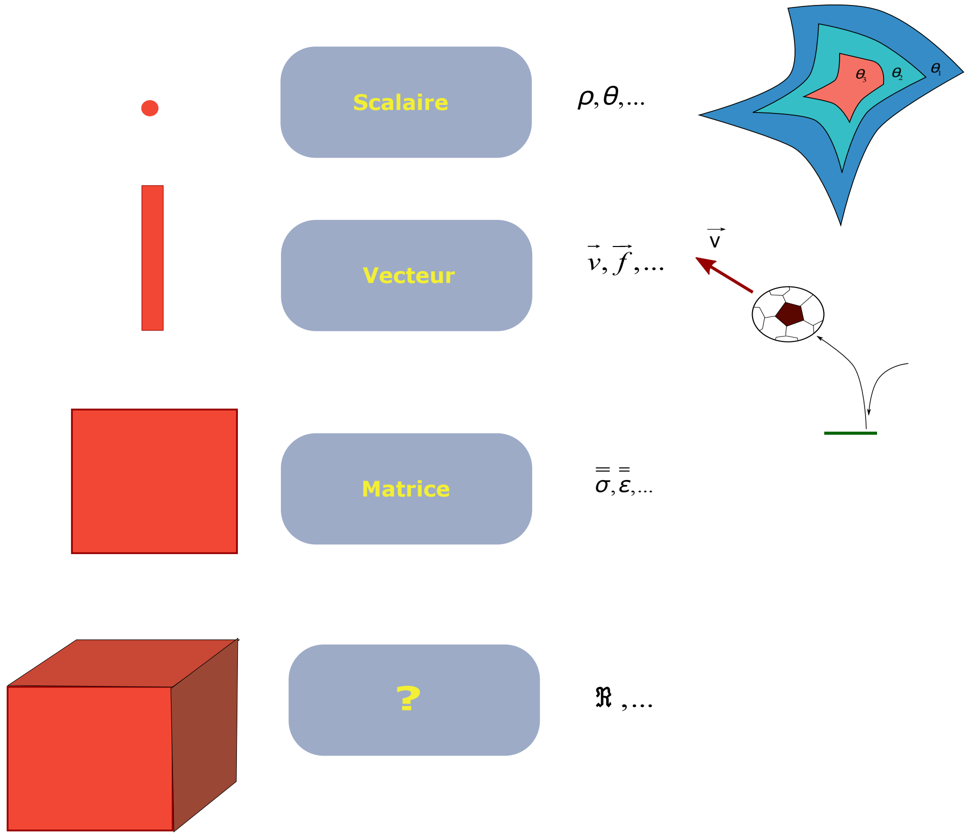
A tensor is an algebraic entity with multiple components that generalizes the concepts of **scalar**, **vector**, and **matrix**.

- Many physical quantities are mathematically represented as tensors.
- Tensors are independent of any reference frame but, for practical purposes, are represented by the components of their matrices.
- The components of a tensor depend on the chosen reference frame and vary with it.

1.1.2 Order of a Tensor

The order of a tensor is determined by the number of indices required to represent it.

- Scalar: zero dimension $\alpha = 3.14$
- Vector: one dimension $\overrightarrow{v} = \begin{pmatrix} 1.2 \\ 0.3 \\ 0.8 \end{pmatrix}$.



- Matrix : 2 dimensions $\overline{\overline{E}} = \begin{pmatrix} 0.1 & 0 & 1.3 \\ 1.6 & 2.4 & 0.5 \\ 0 & 0.4 & 5.8 \end{pmatrix}$.
- Third-order tensor: three dimensions . . .

1.2 Vectors and Tensors

1.2.1 Vector

In a three–dimensional Euclidean space ξ, let $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal basis. A vector $\vec{V}$ is represented by its components V_1, V_2, V_3:

$$\vec{V} = V_1 \vec{e_1} + V_2 \vec{e_2} + V_3 \vec{e_3} = \sum_{i=1}^{3} V_i \vec{e_i} \tag{1.1}$$

Using the summation convention, or **Einstein summation convention**, we write

$$\vec{V} = V_i \vec{e_i} \tag{1.2}$$

where, whenever an index is repeated, it is understood that the index varies from 1 to 3 and the summation is performed. In expression (1.2), the index i is a **dummy index**. In matrix notation, we may also write

$$\vec{V} = \left\{ \vec{V} \right\} = \left\{ \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array} \right\} \tag{1.3}$$

and the transposed vector

$$\vec{V}^T = \left\{ \vec{V} \right\}^T = \left\langle \vec{V} \right\rangle = \left\langle \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \right\rangle \tag{1.4}$$

1.2.2 Linear Mapping from ξ to ξ

Let A be a linear mapping. In the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, this mapping is represented by a 3×3 matrix denoted $[A]$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

If $\vec{W}$ is a vector such that $\vec{W} = A\vec{V}$, then the components of $\vec{W}$ are given by

$$\begin{aligned} W_1 &= A_{11}V_1 + A_{12}V_2 + A_{13}V_3 \\ W_2 &= A_{21}V_1 + A_{22}V_2 + A_{23}V_3 \\ W_3 &= A_{31}V_1 + A_{32}V_2 + A_{33}V_3 \end{aligned}$$

and using the summation convention, where j is a dummy index,

$$W_i = A_{ij}V_j \tag{1.5}$$

and in matrix notation

$$\{W\} = [A]\{V\}$$

The **Kronecker symbol** is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.6}$$

In particular, the identity mapping $\bar{\bar{I}}$ is represented by the matrix

$$\left[\bar{\bar{I}}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

that is, $I_{ij} = \delta_{ij}$. The composition of two linear mappings corresponds to the product of their representative matrices, that is,

$$C = A \circ B \quad \text{or} \quad [C] = [A][B]$$

and in index notation

$$C_{ij} = A_{ik}B_{kj} \tag{1.7}$$

1.2.3 Bilinear Forms

Let A be a bilinear form on ξ, that is, a bilinear mapping from $\xi \times \xi$ into $\mathbf{R}$. In the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ it is represented by a matrix A_{ij} such that

$$A\left(\vec{V}, \vec{W}\right) = A_{ij}V_iW_j \tag{1.8}$$

or in matrix notation

$$A\left(\vec{V}, \vec{W}\right) = \langle V \rangle [A] \{W\}$$

In particular, the bilinear form represented in any basis by the Kronecker symbol corresponds to the dot product. If $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is an orthonormal basis, then

$$e_i \cdot e_j = \delta_{ij}$$

and the dot product of two vectors is given by

$$\vec{V} \cdot \vec{W} = V_i\vec{e_i} \cdot W_j\vec{e_j} = V_iW_j\vec{e_i} \cdot \vec{e_j} = \delta_{ij}V_iW_j = V_iW_i$$

or in matrix notation

$$\vec{V} \cdot \vec{W} = \langle V \rangle \{W\}$$

1.2.4 Tensors

Second-Order Tensor

A second-order tensor T is a linear operator that associates with every vector $\overrightarrow{V}$ in the Euclidean space another vector $\overrightarrow{W}$ in the same space.

$$\overrightarrow{W} = T\left(\overrightarrow{V}\right)$$

This operator can be represented by a 3×3 matrix, denoted $[T]$, $\left[\overline{\overline{T}}\right]$, or simply $\overline{\overline{T}}$, such that

$$W_i = T_{ij}V_j$$

or in matrix notation

$$\left\{\overrightarrow{W}\right\} = \left[\overline{\overline{T}}\right]\left\{\overrightarrow{V}\right\}$$

or

$$\overrightarrow{W} = \overline{\overline{T}}\,\overrightarrow{V}$$

- A tensor is said to be **symmetric** if $T_{ij} = T_{ji}$
- A tensor is said to be **skew-symmetric** if $T_{ij} = -T_{ji}$
- A tensor is said to be **isotropic** if $T_{ij} = t\,\delta_{ij}$
- Any tensor can always be decomposed into a symmetric part and a skew-symmetric part

$$\overline{\overline{A}} = \overline{\overline{A}}^{\text{sym}} + \overline{\overline{A}}^{\text{asym}}, \qquad \overline{\overline{A}}^{\text{sym}} = \frac{1}{2}\left(\overline{\overline{A}} + \overline{\overline{A}}^{T}\right), \qquad \overline{\overline{A}}^{\text{asym}} = \frac{1}{2}\left(\overline{\overline{A}} - \overline{\overline{A}}^{T}\right)$$

Tensor Product

The tensor product of the vector $\overrightarrow{U}$ with the vector $\overrightarrow{V}$, denoted $\overrightarrow{U} \otimes \overrightarrow{V}$, is defined as the second-order tensor associated with the bilinear form that assigns to the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ the quantity $\left(\overrightarrow{U} \cdot \overrightarrow{X}\right)\left(\overrightarrow{V} \cdot \overrightarrow{Y}\right)$. The 9 tensor products $\overrightarrow{e_i} \otimes \overrightarrow{e_j}$ define a basis of the vector space of second-order tensors. Therefore, a tensor T can be written as

$$\overline{\overline{T}} = T_{ij}\,\overrightarrow{e_i} \otimes \overrightarrow{e_j}$$

1.2.5 Change of Reference Frame

Transformation Matrix

Let $\left(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\right)$ be an orthonormal basis and $\left(\overrightarrow{e_1}', \overrightarrow{e_2}', \overrightarrow{e_3}'\right)$ another orthonormal basis. The transformation matrix Q is defined such that

$$\begin{aligned}
\overrightarrow{e_1}' &= Q_{11}\overrightarrow{e_1} + Q_{12}\overrightarrow{e_2} + Q_{13}\overrightarrow{e_3}\\
\overrightarrow{e_2}' &= Q_{21}\overrightarrow{e_1} + Q_{22}\overrightarrow{e_2} + Q_{23}\overrightarrow{e_3}\\
\overrightarrow{e_3}' &= Q_{31}\overrightarrow{e_1} + Q_{32}\overrightarrow{e_2} + Q_{33}\overrightarrow{e_3}
\end{aligned}$$

or, using index notation,

$$\overrightarrow{e_i}' = Q_{ij}\overrightarrow{e_j}$$

and in matrix notation

$$\left\{\overrightarrow{e'}\right\} = [Q]\left\{\overrightarrow{e}\right\}$$

Since the two bases are orthonormal, we must have

$$\delta_{ij} = \overrightarrow{e_i}' \cdot \overrightarrow{e_j}' = Q_{ik}\overrightarrow{e_k} \cdot Q_{jl}\overrightarrow{e_l} = Q_{ik}Q_{jl}\delta_{kl} = Q_{ik}Q_{jk}$$

which shows that the inverse matrix of Q is Q^T.

Vectors

Let $\overrightarrow{V}$ be a vector with components V_i in the basis $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ and V_i' in the basis $\left(\overrightarrow{e_1'}, \overrightarrow{e_2'}, \overrightarrow{e_3'}\right)$.

$$\overrightarrow{V} = V_i \overrightarrow{e_i} = V_i' \overrightarrow{e_i'}$$

Using the transformation matrix,

$$\overrightarrow{V} = V_i \overrightarrow{e_i} = V_i Q_{ki} \overrightarrow{e_k}$$

hence

$$V_k' = V_i Q_{ki} \quad \text{and} \quad V_k = V_i' Q_{ik}$$

or, in matrix notation,

$$\left\{\overrightarrow{V'}\right\} = [Q]\left\{\overrightarrow{V}\right\} \quad \text{and} \quad \left\{\overrightarrow{V}\right\} = [Q]^T\left\{\overrightarrow{V'}\right\}$$

Remark: the dot product is an invariant, that is, this quantity is independent of the chosen reference frame. In index notation,

$$\overrightarrow{V'} \cdot \overrightarrow{W'} = V_k' W_k' = V_i Q_{ki} W_j Q_{kj} = \delta_{ij} V_i W_j = V_i W_i = \overrightarrow{V} \cdot \overrightarrow{W}$$

and in matrix notation

$$\overrightarrow{V'} \cdot \overrightarrow{W'} = \left\langle\overrightarrow{V'}\right\rangle\left\{\overrightarrow{W'}\right\} = \left([Q]\left\{\overrightarrow{V}\right\}\right)^T [Q]\left\{\overrightarrow{W}\right\} = \left\langle\overrightarrow{V}\right\rangle [Q]^T [Q]\left\{\overrightarrow{W}\right\} = \left\langle\overrightarrow{V}\right\rangle\left\{\overrightarrow{W}\right\} = \overrightarrow{V} \cdot \overrightarrow{W}$$

Linear Mapping

Let A be a linear mapping with components A_{ij} in the basis $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ and A_{ij}' in the basis $\left(\overrightarrow{e_1'}, \overrightarrow{e_2'}, \overrightarrow{e_3'}\right)$. In index notation,

$$W_i' = A_{ik}' V_k' = Q_{ij} W_j = Q_{ij} A_{jm} V_m = Q_{ij} A_{jm} Q_{km} V_k'$$

from which

$$A_{ik}' = Q_{ij} A_{jm} Q_{km}$$

and in matrix notation

$$\left\{\overrightarrow{W'}\right\} = [A']\left\{\overrightarrow{V'}\right\} = [Q]\left\{\overrightarrow{W}\right\} = [Q][A]\left\{\overrightarrow{V}\right\} = [Q][A][Q]^T\left\{\overrightarrow{V'}\right\}$$

hence

$$[A'] = [Q][A][Q]^T$$

1.3 Permutations and Determinants

1.3.1 Permutation Symbols (Levi-Civita Symbol)

Let i, j, and k be three indices with different values. They are said to form an even permutation of $1,2,3$ if they can be brought into this order by an even number of permutations. They form an odd permutation of $1,2,3$ if they can be brought into this order by an odd number of permutations. The even permutations of $1,2,3$ are therefore: $(1,2,3)$, $(3,1,2)$, and $(2,3,1)$, while the odd permutations are: $(2,1,3)$, $(1,3,2)$, and $(3,2,1)$. The permutation symbol, known as the **Levi-Civita symbol**, is defined by ε_{ijk} such that

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if} \quad \text{two indices are equal} \\ +1 & \text{if} \quad i,j,k \text{ form an even permutation} \\ -1 & \text{if} \quad i,j,k \text{ form an odd permutation} \end{cases} \tag{1.9}$$

These symbols represent the scalar triple product of the basis vectors

$$\varepsilon_{ijk} = (\vec{e_i}, \vec{e_j}, \vec{e_k})$$

The quantities ε_{ijk} are the components of a third-order tensor that represents, for example, the trilinear form known as the scalar triple product:

$$\left(\vec{U}, \vec{V}, \vec{W}\right) = \vec{U} \cdot \left(\vec{V} \wedge \vec{W}\right) = \varepsilon_{ijk} U_i V_j W_k$$

1.3.2 Determinant of a Matrix

The permutation symbols allow the computation of the determinant of a matrix as

$$\varepsilon_{ijk}\,\mathrm{Det}(A) = \varepsilon_{mnp} A_{im} A_{jn} A_{kp} \tag{1.10}$$

or equivalently

$$\mathrm{Det}(A) = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{mnp} A_{im} A_{jn} A_{kp}$$

1.3.3 Adjoint of an Antisymmetric Tensor

Let Ω be an antisymmetric tensor

$$\Omega = \begin{bmatrix} 0 & \Omega_{12} & -\Omega_{31} \\ -\Omega_{12} & 0 & \Omega_{23} \\ \Omega_{31} & -\Omega_{23} & 0 \end{bmatrix}$$

It is also possible to associate with it the vector

$$\vec{\omega} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \Omega_{23} \\ \Omega_{31} \\ \Omega_{12} \end{Bmatrix}$$

so that

$$\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

The vector $\vec{\omega}$ is the adjoint vector of the antisymmetric tensor Ω.
In index notation we have

$$\left\{ \begin{array}{l} \Omega_{ij} = \varepsilon_{ijk}\,\omega_k \\ \omega_i = \frac{1}{2} \varepsilon_{ijk}\,\Omega_{jk} \end{array} \right. \tag{1.11}$$

1.4 Vector Calculus and Vector Analysis

1.4.1 Vector Calculus

The cross product

$$\vec{c} = \vec{a} \wedge \vec{b}$$

can be written in index notation as

$$c_i \vec{e_i} = \varepsilon_{ijk} a_j b_k \vec{e_i}$$

It can be shown that

$$(\vec{a} \wedge \vec{b}) \wedge \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$
$$(\vec{a} \wedge \vec{b}) \cdot (\vec{c} \wedge \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

1.4.2 Vector Analysis

A comma denotes partial differentiation, i.e. $,i = \frac{\partial}{\partial x_i}$. All operators presented in this section are expressed in an orthonormal Cartesian coordinate system.

Scalar Function

The gradient of a scalar function is a vector

$$\overrightarrow{grad} f = \nabla f = f_{,i}\,\overrightarrow{e_i} = \left\{ \begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{array} \right\}$$

The Laplacian of a scalar function is a scalar

$$\Delta f = f_{,ii} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Vector Field

The divergence of a vector is a scalar

$$Div\,\overrightarrow{v} = v_{i,i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

The curl of a vector is a vector

$$\overrightarrow{rot}\,\overrightarrow{v} = \nabla \wedge \overrightarrow{v} = \varepsilon_{ijk} v_{k,j}\,\overrightarrow{e_i} = \left\{ \begin{array}{c} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{array} \right\}$$

The gradient of a vector is a matrix

$$\nabla \overrightarrow{v} = v_{i,j}\,\overrightarrow{e_i} \otimes \overrightarrow{e_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Second-Order Tensor

The divergence of a tensor is a vector

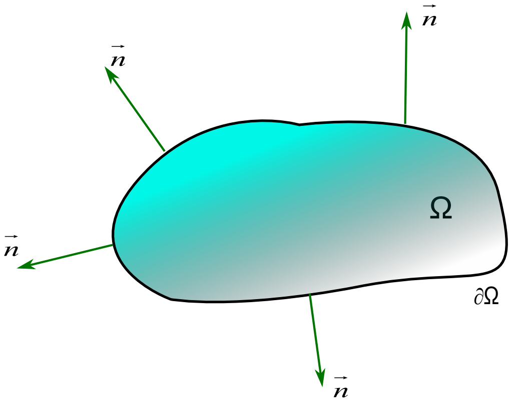
$$Div\overline{\overline{T}} = T_{ij,j}\,\overrightarrow{e_i} = \left\{ \begin{array}{c} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{array} \right\}$$

1.4.3 Integral Transformation

Integral transformation formulas are mathematical tools that allow the dimension of an integration domain to be modified. The most commonly used are the **Ostrogradsky formulas**, which establish a correspondence between an integral over a domain D (volume integral) and an integral over its boundary S (surface integral).

Let Ω be a bounded domain and $\partial\Omega$ its boundary with outward normal $\overrightarrow{n}$. Let ϕ be a scalar function. Then

$$\iint_{\partial\Omega} \phi\,\overrightarrow{n}\,dS = \iiint_{\Omega} \overrightarrow{grad}\phi\,dV$$



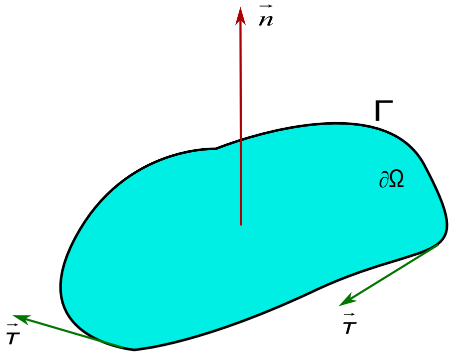
Let $\overrightarrow{A}$ be a vector. Then

$$\iint_{\partial\Omega} \overrightarrow{A} \cdot \overrightarrow{n}\, dS = \iiint_{\Omega} Div(\overrightarrow{A})\, dV$$

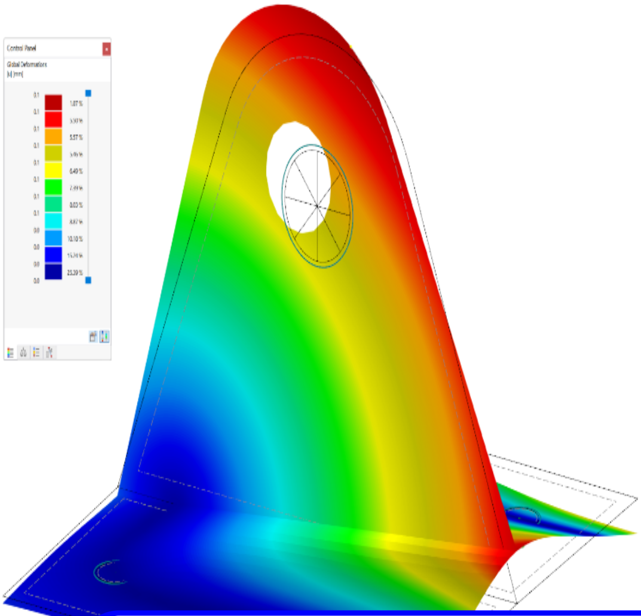
Let $\overline{\overline{T}}$ be a tensor. Then

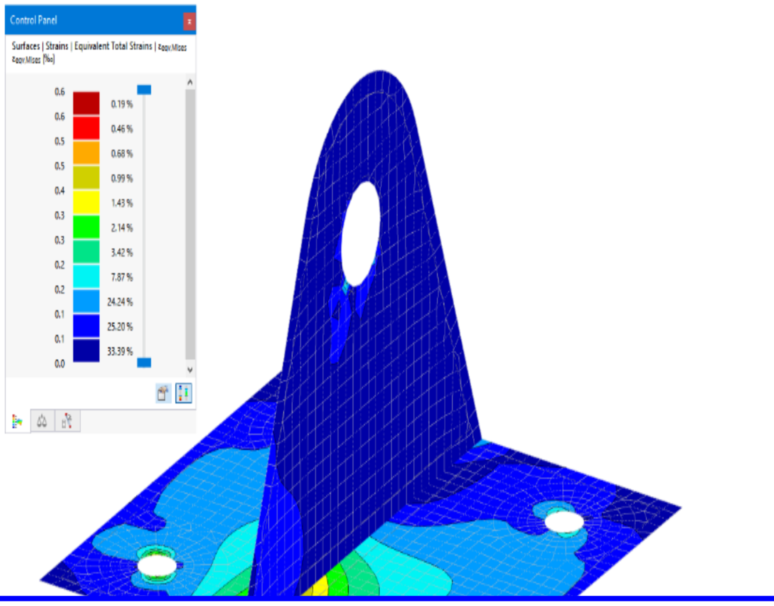
$$\iint_{\partial\Omega} \overline{\overline{T}} \cdot \overrightarrow{n}\, dS = \iiint_{\Omega} Div(\overline{\overline{T}})\, dV$$

Let $\partial\Omega$ be a planar domain with normal $\overrightarrow{n}$ and boundary Γ. Let $\overrightarrow{U}$ be a vector defined on this domain. If $\overrightarrow{\tau}$ is the unit tangent vector to Γ, then



$$\iint_{\partial\Omega} rot(\overrightarrow{U}) \cdot \overrightarrow{n}\, dS = \int_{\Gamma} \overrightarrow{U} \cdot \overrightarrow{\tau}\, dl$$





2. Introduction to Continuum Mechanics

2.1 Some Definitions

- **Mechanics.** This is the branch of physics that describes the motions and equilibrium states of a system.
- **Medium.** We do not speak here of a material but of a medium. We are therefore dealing with an "abstract" theory which, at first, is somewhat removed from physical and experimental reality. This allows the theory to remain as general as possible. It will only later be specialized into different branches dealing with different materials.
- **Continuum.** The notion of continuity is closely related to that of scale. We are going to present an abstract theory, but we must keep in mind that it is intended to be applied to the physical world. All real materials are discontinuous at small scales (molecules, crystals, grains of sand, etc.). A real material will therefore be considered continuous (this notion will be clarified later) provided that one looks from sufficiently far away, or with sufficiently limited resolution.

More generally, **Continuum Mechanics (CM)** is a mathematical construction. It is a modeling framework. In other words, it is an abstract representation of the physical world based on assumptions that are more or less valid depending on the application. As always in the physical sciences, the ultimate criterion for determining whether the theory is valid in a given case is well known: experimentation.

2.2 The Place of Continuum Mechanics within Mechanics

The simplest of the "modern" branches of mechanics is point mechanics (based on Newton's results). It makes it possible to determine the position, velocity, and acceleration of a "material point," that is, a mathematical object with no "shape" or "volume," but endowed with mass and subjected to forces. This is the theory used, for example, to study the motion of planets in their orbits.

Sir Isaac Newton
1643 - 1727

R Isaac Newton, born in 1643, was an English, and later British, philosopher, mathematician, physicist, alchemist, astronomer, and theologian. An iconic figure in science, he is best known for laying the foundations of classical mechanics, for his theory of universal gravitation, and for its formulation. Newton showed that the motion of objects on Earth and of celestial bodies is governed by the same natural laws; building on Kepler's laws of planetary motion, he developed the *universal law of gravitation*.

The next stage is rigid body mechanics. It retains the ingredients of point mechanics, but adds to the system under study a "shape," and therefore a volume and a spatial distribution of mass. This branch of mechanics introduces the notions of rotation, inertia, and moment. It applies very well, for example, to the study of articulated systems (robotics, etc.) or to the micromechanical study of sand.

The essential addition brought by **Continuum Mechanics** is the possibility for the system to deform. Of course, this considerably complicates the theory, and in particular requires much more sophisticated mathematical tools than those needed for the first two branches of mechanics.

2.3 Main Assumptions of Continuum Mechanics

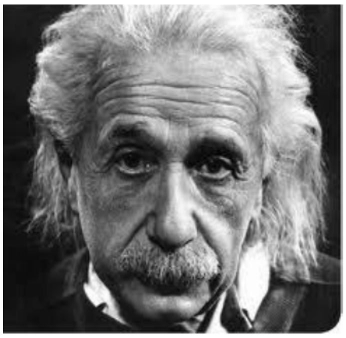
Continuum Mechanics is a so-called "classical" mechanics, which means that:

- the scale of the problem is much larger than the size of the elementary particles. Continuum Mechanics is therefore not **quantum mechanics**.

Max Planck
1858 - 1947

R Max Planck, born Max Karl Ernst Ludwig Planck in 1858 in Kiel and deceased in 1947 in Göttingen, was a German physicist. He was awarded the Nobel Prize in Physics in 1918 for his work on quantum theory. Max Planck was one of the founders of *quantum mechanics*. His work also led to the concept of the Planck era, a period in the history of the Universe during which the four fundamental interactions were unified.

- the speed of matter is much smaller than the speed of light. Continuum Mechanics is therefore not **relativistic**.



Albert Einstein
1879 - 1955

R Albert Einstein, born in 1879 in Ulm and deceased in 1955 in Princeton, was a theoretical physicist. He successively held German nationality, became stateless in 1896, then Swiss in 1901, and finally held dual Swiss-American nationality from 1940. He published his theory of special relativity in 1905 and his theory of gravitation, known as general relativity, in 1915. He also contributed greatly to the development of quantum mechanics and cosmology, and received the Nobel Prize in Physics in 1921 for his explanation of the photoelectric effect. In popular culture, his name and image are directly associated with intelligence and knowledge.

The main assumption of **Continuum Mechanics** is called the continuity assumption. It states that the properties of matter (density, mechanical properties, etc.) are continuous. This implies that they can be described by continuous and differentiable mathematical objects (at least piecewise). These objects will be called **fields**, and they will depend on both space and time.

More generally, this means that when **Continuum Mechanics** is applied to the real world, it deals with local averages. It is impossible to determine the motion of each elementary particle that makes up a physical system (for example, one cubic millimeter of ambient air contains approximately 10^{18} molecules), so one must consider a sufficiently large volume for it to be treated as continuous. For instance, the pressure exerted by a fluid on a solid wall results from a multitude of impacts of fluid molecules on the solid, yet it is represented by a single scalar quantity, corresponding to the average mechanical effect of these impacts over a small time interval and over a small area of the solid.

Later on, a number of additional assumptions will gradually be introduced into the theory (small perturbation assumption, conservation of mass, fundamental principle of dynamics, etc.) whenever it becomes clear that the theory cannot progress without them.

2.4 Typology of Continuous Media

Matter as we know it in the physical world is often considered to exist in one of the three well-known states: solid, liquid, and gas. Under various conditions, these three states may all be regarded as continuous media, although this is less straightforward for the gaseous state, which is of limited interest to geotechnical engineers anyway.

One may also consider a large number of intermediate states: fluid, pasty, soft, thick, plastic, viscous, and so on. For example, a piece of raclette cheese subjected to the thermal action of a melting pan gradually changes from solid to liquid, yet it is difficult to define precisely the exact moment at which the transition occurs. This shows that the boundaries between the three classical states of matter are in fact rather blurred.

Continuum Mechanics stands above these considerations. At first, we therefore set aside any notion of "consistency" and propose a mathematical model of the deformation of a medium that ignores this notion completely. Only later will we introduce the concept of a constitutive model and formulate mathematical models, based on experiments and/or physical considerations, that are appropriate for each application in the physical world.

2.5 Applications of Continuum Mechanics

As a theory of the deformation of matter, **Continuum Mechanics** is absolutely ubiquitous in engineering. Even though not all engineers can claim to master it completely, the vast majority make use of it, sometimes unknowingly, at least through mechanical simulation software.

Among the most obvious applications are:

- Industrial processes (sheet forming, machining, etc.), the mechanical industry (automotive, machine tools, aeronautics, etc.), biomechanics, composite materials, micromechanics, mechatronics, and so on.
- Civil engineering structures, buildings, bridges, dams, roads, made of concrete, steel, wood, aluminum, and so on.
- Fluid mechanics, aerodynamics, flows in channels and pipes, river flows and groundwater flows, and so on.
- Geophysics, soil mechanics, rock mechanics, and so on.
- Many other scientific and technical disciplines.

Continuum Mechanics is therefore one of the main common languages shared by engineers. In the first stages of its formulation, as in these lecture notes, it has the status of a science. In its many applications, it then branches into numerous engineering techniques specific to each discipline.

For the geotechnical engineer, **Continuum Mechanics**, coupled with related disciplines such as soil mechanics and fluid mechanics, is directly or indirectly involved when addressing questions such as:

- By how many millimeters will my foundation settle under this load?
- After how much time will the final settlement be reached?
- What load can this retaining structure support before failure?
- Will this slope remain stable during heavy rainfall?
- What water discharge will seep through this dam?

These are only a few examples among many. In general, **Continuum Mechanics** comes into play whenever one has to deal with problems of stiffness, strength, stability, or flow.

2.6 Spirit of the Course

Most of the equations presented in this course are mathematically very complex. For the vast majority of them, it is impossible to solve them analytically by hand. Moreover, within the framework of this course, that is not necessary. Our primary objective in this introduction to **Continuum Mechanics** is to formulate equations, not to solve them.

Of course, engineers have an entire arsenal of numerical methods available to perform such solutions, which is rather reassuring regarding the usefulness of the theory. Nevertheless, in order to fully benefit from this course, it is better to set aside the notion of solution at first. We will focus much more on the theory and on the fundamental reasons why such systems of equations must be formulated.

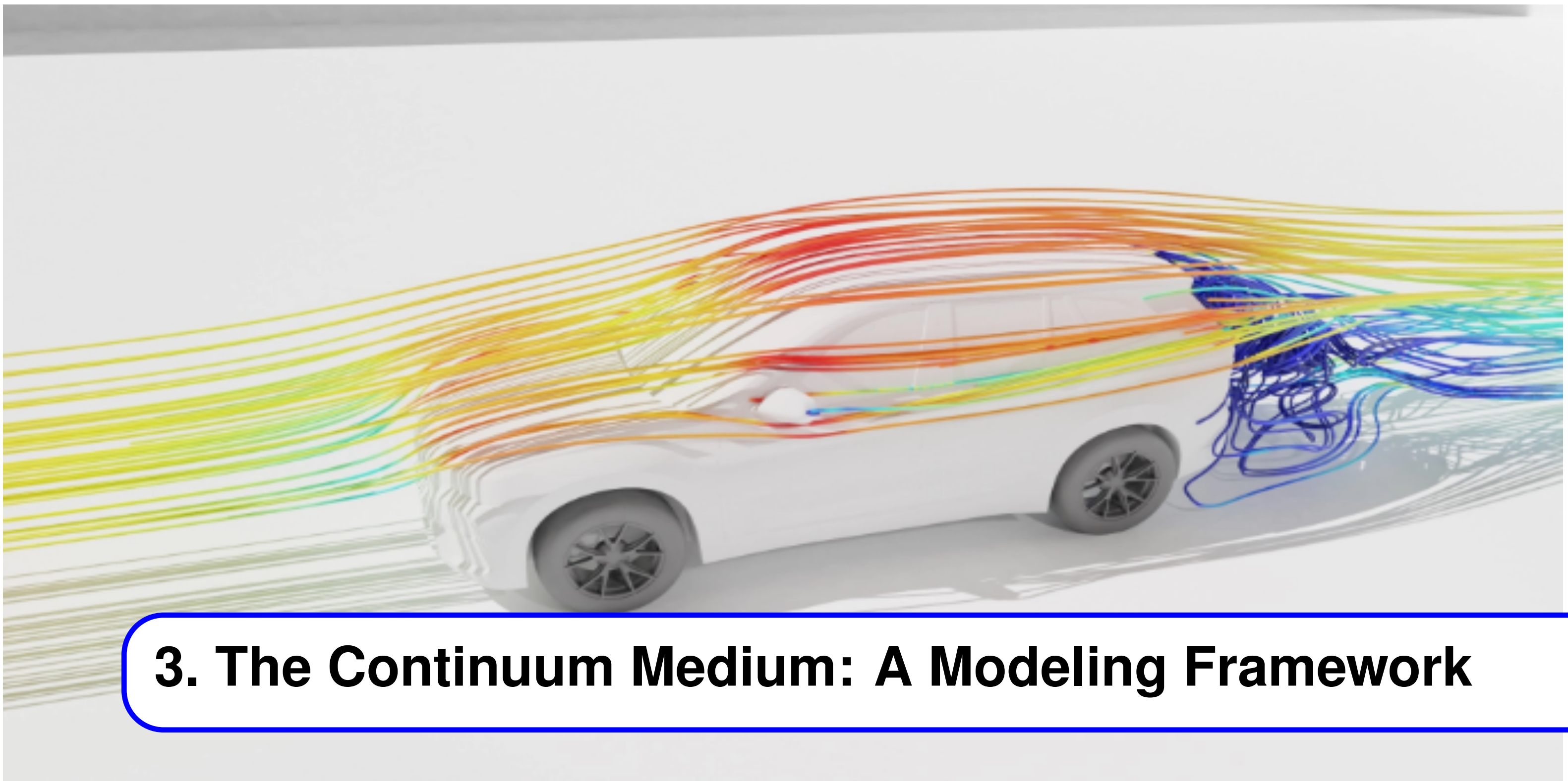
II

CINÉMATIQUE

3 The Continuum Medium: A Modeling Framework . . . 35
3.1 Scale, Modeling, Validation
3.2 Concepts and Their Formulation
3.3 Lagrangian and Eulerian Points of View
3.4 Lagrangian Description
3.5 Eulerian Description
3.6 Exercise

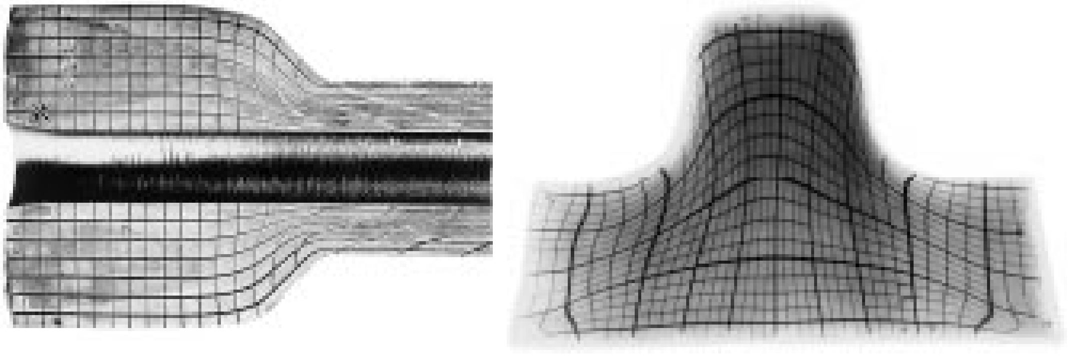
4 Study of the Deformation of a Continuum Medium . . . 51
4.1 Introduction
4.2 Gradient of a Transformation
4.3 Stretch Tensor
4.4 Strain Tensors
4.5 Formulation in Terms of Displacements
4.6 Small-Strain Assumption
4.7 Small-Transformation Assumption
4.8 Compatibility Conditions
4.9 Exercise

5 Kinematics of the Continuum Medium . 75
5.1 Introduction
5.2 Velocity Gradient Tensor
5.3 Strain-Rate Tensor
5.4 Rotation-Rate Tensor
5.5 Material Derivative
5.6 Exercise



3. The Continuum Medium: A Modeling Framework

3.1 Scale, Modeling, Validation



(a) Tube extrusion.

(b) Forging of an alloy billet

Figure 3.1: Cold or hot forming

It is from the observation of the deformation of a solid, for example during a cold or hot forming operation (Figures 3.1a and 3.1b), of the flow of a liquid, or of the expansion or compression of a gas, that the notion of a deformable continuum medium originates. It means that the observer draws from these experiments the idea that certain problems can be treated at a **macroscopic** scale by assimilating matter to a “continuous” medium, without contradicting the models of microscopic physics.
The notion of a relevant ***scale*** for a given problem is thus introduced: although obviously related to the phenomena involved, it depends essentially on the nature of the questions being asked about them. An example taken from the daily practice of some engineers helps illustrate this point.

In this chapter, we aim to clarify and then formulate mathematically the concept of continuity.

We will thus present the modeling of the classical three-dimensional continuum medium, whose applications concern both the mechanics of deformable solids and fluid mechanics (this traditional distinction even becomes difficult to maintain in some cases, for example in the deformation or flow of polymers).

3.2 Concepts and Their Formulation

3.2.1 The Guiding Idea

In the usual experimental approach, whether one follows the flow of a fluid or the deformation of a solid, the observer is led to mark material elements constituting the studied system at a given instant and then track their geometric evolution (Figures 3.1a and 3.1b). It is clear that the marking of such an element, however fine it may be, concerns a "small material domain," regarded as infinitesimal at the macroscopic scale of the mechanician, but still lying above the microscopic scale of the physicist.

The intuitive concept of continuity refers to the evolution over time of the geometric positions of these marked elements: material elements that are neighbors at a given instant remain neighbors over time, and their evolutions are comparable. Clearly, the validity of this concept depends first and foremost on the actual possibility of identifying the relevant small material domains.

The mathematical modeling of the physical concept of continuity is therefore, first of all, geometric.

From the geometric point of view, the modeling of the classical continuum medium starts from the idea that the mechanical system under consideration, S, is represented by a volume Ω whose "$d\Omega$" represent the elementary constituents called "particles."

Since the concept of continuity is related to the observation of ***the evolution of the system***, one wishes to ensure, during this evolution, for neighboring particles:

- preservation of geometric proximity,
- comparable evolution of physical properties.

3.2.2 Configurations of the System

The state of the system S at time t in a reference frame R is called the **configuration of the system**. The current configuration (current time t) is denoted generically by κ_t. The geometric configuration of S is described by the set of positions, identified in the reference frame R, of its particles. This description of the geometric configuration in R is carried out by means of the position vector $\overrightarrow{OM}$, also denoted $\overrightarrow{x}$, which specifies the position of each particle of S at time t from the origin O of a frame R. One may define $\overrightarrow{x}$ by its coordinates (x,y,z) or $x^i(i=1,2,3)$ in R. The volume occupied by S in this configuration is Ω_t with boundary $\partial\Omega_t$.

One also introduces the notion of a **reference configuration**: this is the particular configuration κ_0 of the system at a fixed time t_0. Unless explicitly stated otherwise, we set $t_0 = 0$ (and the reference configuration may also be called the initial configuration). In κ_0, S occupies a volume Ω_0 with boundary $\partial\Omega_0$ (Figure 3.2). The coordinates of the position vectors $\overrightarrow{OM_0}$ in κ_0 will systematically be denoted (X,Y,Z) or $X^i(i=1,2,3)$. The vector $\overrightarrow{OM_0}$ will also be denoted $\overrightarrow{X}$.

Thus:

$$\left\{\begin{array}{llllll} \overrightarrow{OM} = \overrightarrow{x} & \text{with coordinates} & (x,y,z) & \text{or} & x^i & \text{in} \quad \kappa_t \\ \overrightarrow{OM_0} = \overrightarrow{X} & \text{with coordinates} & (X,Y,Z) & \text{or} & X^i & \text{in} \quad \kappa_0 \end{array}\right. \tag{3.1}$$

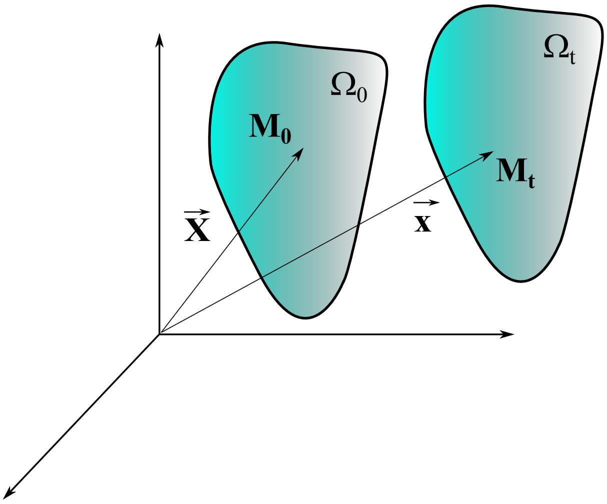


Figure 3.2: Configurations of the system.

3.3 Lagrangian and Eulerian Points of View

Two points of view coexist when it comes to describing motion. These are called the Lagrangian and Eulerian descriptions of motion. The first is more suited to solid mechanics, and the second to fluid mechanics, although this distinction is by no means rigid. In reality, each of these descriptions provides tools that simplify, to varying extents, the definition and solution of a problem depending on the context. It is important to understand both.

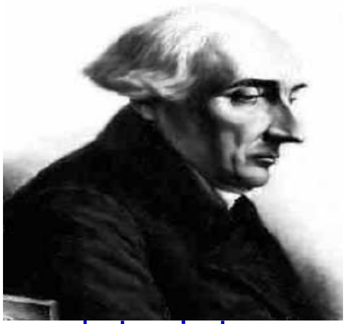
3.4 Lagrangian Description

3.4.1 Definition

We now aim to provide a precise mathematical formulation of the intuitive ideas and concepts presented in the previous sections.

In this spirit, referring to the marking of small material domains mentioned in Paragraph 2.1, the Lagrangian description consists in:

- identifying the constituent particles of the system by their geometric position in a configuration taken as reference and denoted κ_0, that is, by the vector variable $\overrightarrow{X}$,
- expressing the value of any physical quantity in the current configuration as a function of the particle to which it is attached and the current time, that is, as a function of the variables $\overrightarrow{X}$ and t.



Joseph-Louis Lagrange
1736 - 1813

R Joseph-Louis, Comte de Lagrange, born in Turin in 1736 and deceased in Paris in 1813, was an Italian mathematician, mechanician, and astronomer who later became a naturalized French citizen. At the age of thirty, he left Piedmont and lived in Berlin for twenty-one years. He then settled in Paris for the last twenty-six years of his life, where he obtained French nationality at the request of Antoine Lavoisier.

Thus, the position vector $\overrightarrow{OM} = \overrightarrow{x}$ of the particle initially located at M_0 in κ_0 is given by

$$\overrightarrow{x} = \overrightarrow{\phi}(\overrightarrow{X},t) \tag{3.2}$$

and the value of a physical quantity attached to this particle, say B, is

$$B = B\left(\overrightarrow{X},t\right) \tag{3.3}$$

In Formula 3.2, $\overrightarrow{\phi}$ is a vector-valued function defined on Ω_0, for all t, and which obviously satisfies

$$\overrightarrow{\phi}\left(\overrightarrow{X},0\right) = \overrightarrow{X} \tag{3.4}$$

while, depending on the nature of the physical quantity under consideration, B is a scalar, vector, or tensor-valued function of arbitrary order that satisfies the analogous formula to 3.4.

The function $\overrightarrow{\phi}$ thus describes the geometric correspondence between the configurations γ_0 and γ_t. As a function of t, it defines the entire geometric evolution of the system S, that is, its motion.

3.4.2 Continuity Assumptions

We now naturally examine the mathematical conditions on $\overrightarrow{\phi}$ that properly account for the intuitive concept of continuity introduced above.

We propose the following assumptions:

- $\overrightarrow{\phi}$ is a bijection from Ω_0 onto Ω_t, whose inverse bijection is denoted by $\overrightarrow{\psi}$:

$$\overset{\forall t, \forall M_0}{\overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{X},t\right)} \quad \Leftrightarrow \quad \overset{\forall t, \forall M}{\overrightarrow{X} = \overrightarrow{\psi}(\overrightarrow{x},t)} \tag{3.5}$$

- $\overrightarrow{\phi}$ and $\overrightarrow{\psi}$ are continuous with respect to all **space and time variables**.

These assumptions have the following classical consequences, which make it possible to assess the validity of this modeling by comparison with experiments.

1. Two particles occupying “infinitely close” positions in κ_0 remain infinitely close in any configuration.
2. Particles occupying a connected domain in κ_0 occupy, in κ_t, a connected domain of the same order (volume, surface, curve). This allows the mathematical definition of the concept of a **material domain**: a domain **transported by the motion**, that is, at each instant the geometric domain occupied by the same set of particles.
3. Particles located inside a closed surface in κ_0 remain, at every instant t, inside the transported surface. Thus, the boundary of a material volume is a material surface.
4. In particular, the boundary of S is a material surface, which means that it is always made up of the same particles.

5. As a consequence of the assumptions of continuous differentiability, let $J\left(\overrightarrow{X},t\right)$ be the Jacobian determinant of $\overrightarrow{\phi}$ at time t at (X_1,X_2,X_3), i.e. the determinant of the Jacobian matrix of the first derivatives of x_i with respect to X_i:

$$J\left(\overrightarrow{X},t\right)=\frac{D(x_1,x_2,x_3)}{D(X_1,X_2,X_3)} \tag{3.6}$$

Since $\overrightarrow{\phi}$ is continuous and continuously differentiable, as is $\overrightarrow{\psi}$, it follows that $J\left(\overrightarrow{X},t\right)$ is continuous with respect to X and t. Moreover, it can be neither zero nor infinite, since the Jacobian matrices of $\overrightarrow{\phi}$ and $\overrightarrow{\psi}$ must be invertible. It therefore preserves a constant sign over Ω_0 and throughout the motion. It then follows from 3.4, where one has

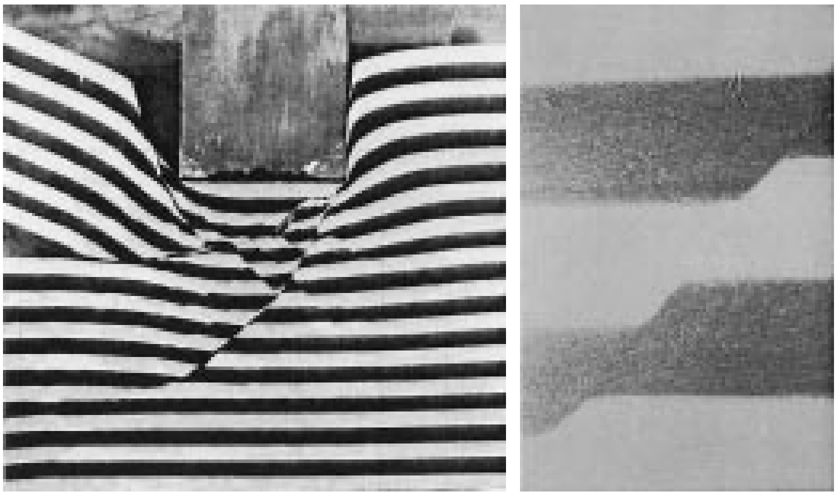
$$J\left(\overrightarrow{X},0\right)=1 \quad \forall M_0\in\Omega_0 \tag{3.7}$$

that $J\left(\overrightarrow{X},t\right)$ is positive and finite for all $M_0\in\Omega_0$, for all t:

$$0<J\left(\overrightarrow{X},t\right)<+\infty \tag{3.8}$$

3.4.3 Relevance of the Model: Weakening of the Continuity Assumptions

The consequences stated above appear consistent with the intuition of continuity that is at the very origin of the modeling. However, one is led to moderate the mathematical assumptions somewhat in order to deal more conveniently with certain observed phenomena.



(a) (b)

Figure 3.3: Asymmetric punching of a plasticine block

These include, for example, cracks encountered in fracture mechanics, failure surfaces, slip surfaces, localization of deformation in solid mechanics (Figures 3.3a and 3.3b), and jet surfaces in fluid mechanics (Figure 3.4). For these cases, preserving the proximity of two initially neighboring points throughout the evolution is too restrictive: one must allow discontinuities of ϕ when crossing certain surfaces.

For these reasons, it is agreed to weaken the continuity assumptions by requiring only **piecewise continuity and piecewise continuous differentiability of ϕ**: discontinuities of the function ϕ and/or its derivatives are allowed across a countable infinity of surfaces in R^3.

Figure 3.4: Flow around a flat plate.

3.4.4 Physical Interpretation of the Lagrangian Description: Trajectories

The Lagrangian description is the mathematical formulation of a simple experimental reality. Indeed, Formula (3.2) describes the motion of each particle of the system: if one considers the particle identified by $\overrightarrow{X}$, (3.2) provides the description of its trajectory, parameterized with respect to time, in the reference frame R:

$$\overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{X}, t\right) \quad \text{where} \quad \overrightarrow{X} \quad \text{is fixed}$$

For this reason, the Lagrangian description is also called a “trajectory-based” description.

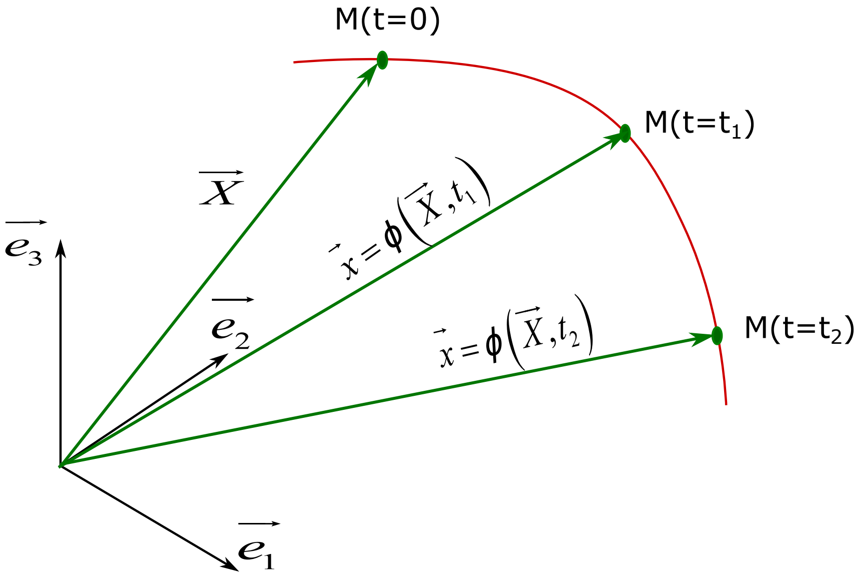


Figure 3.5: Trajectory of a particle.

In practice, visualizing the trajectory of a particle from a given instant t_0 is achieved by marking a particle at time t_0, then taking a long-exposure view of the motion of the system from that instant onward. This type of experiment is commonly carried out in both solid mechanics and fluid mechanics.

3.4.5 Streaklines

Taking an **instantaneous snapshot** leads to the introduction of another type of geometric curve, namely **streaklines**, defined as follows (Figure 9).

At a **geometric** point P in R, with coordinates x_P^i, and starting from time t_0, one marks each particle passing through P; at time $T > t_0$, one observes the positions of these particles in R: the corresponding **geometric curve** is the **streakline of point** P observed at time T.

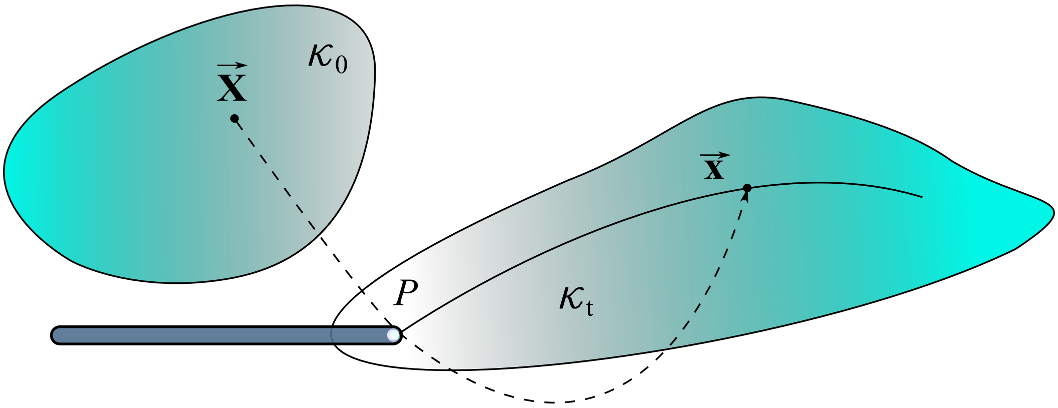


Figure 3.6: Streakline (dashed)

The equation of this curve, parameterized by t', is obtained from (3.2) and (3.5) by following the particle $\vec{X}$ which, passing through P at time t_0, has been marked and is located at $\vec{x}$ at time T on the streakline:

$$\vec{x} = \vec{\phi}\left(\vec{\psi}\left(\vec{x}_P, t'\right), T\right) \quad t_0 \leq t' \leq T \tag{3.9}$$

An example of visualization of streaklines is shown in Figure 3.7.

3.4.6 Velocity of a Particle

The velocity, in the reference frame R, of the particle identified by its position $\vec{X}$ in κ_0 is obtained directly from (3.2). It is the vector

$$\vec{v}\left(\vec{X}, t\right) = \frac{\partial \vec{\phi}\left(\vec{X}, t\right)}{\partial t} \tag{3.10}$$

The velocity vector $\vec{v}\left(\vec{X}, t\right)$ in R is obviously tangent to the trajectory of the particle in R at the point $\vec{x} = \vec{\phi}\left(\vec{X}, t\right)$.

Similarly, the acceleration of the particle is written as

$$\vec{\gamma}\left(\vec{X}, t\right) = \frac{\partial^2 \vec{\phi}\left(\vec{X}, t\right)}{\partial t^2} \tag{3.11}$$

3.5 Eulerian Description

3.5.1 Definition

Referring to the physical interpretation of the Lagrangian description based on long-exposure images, the Eulerian description can be introduced through the following intuitive geometric idea: the

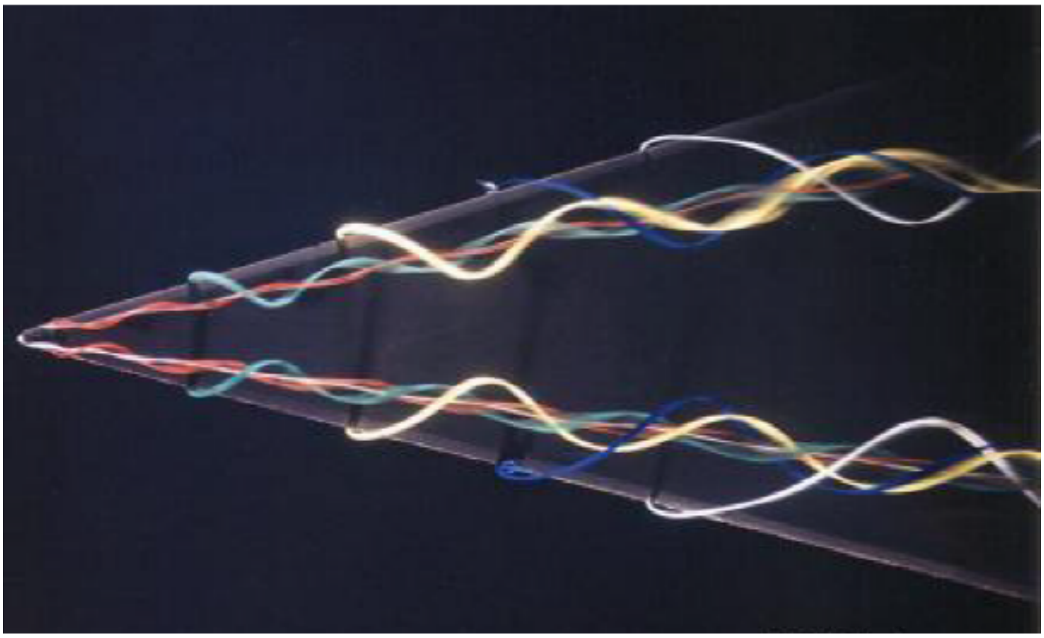


Figure 3.7: Visualization of streaklines originating from the leading edge of a delta wing using dye injection. (ONERA image.)

image obtained by long exposure can be reconstructed by superposing a succession of instantaneous snapshots.

In a more mathematical formulation, the Eulerian description of evolution consists in taking the current configuration at each instant as the reference configuration in order to describe the infinitesimal evolution between t and $(t+dt)$.

Leonard Euler
1707 - 1783

R Leonhard Euler, born in Basel in 1707 and deceased in Saint Petersburg in 1783, was a Swiss mathematician and physicist who spent most of his life in the Russian Empire and in Germany. Euler is regarded as one of the leading mathematicians of the 18^{th} century and one of the greatest and most prolific mathematicians of all time. A statement attributed to Pierre-Simon de Laplace expresses Euler's influence on mathematics: "*Read Euler, read Euler, he is the master of us all.*"

Thus, from the geometric point of view, the Eulerian description defines the motion of the system by giving, at each time t, the velocity $\vec{v_t}$ of the particle located at the geometric point M in κ_t:

$$\forall t, \forall M \in \Omega_t, \quad \vec{v} = \vec{v_t}\left(\vec{x},t\right) \tag{3.12}$$

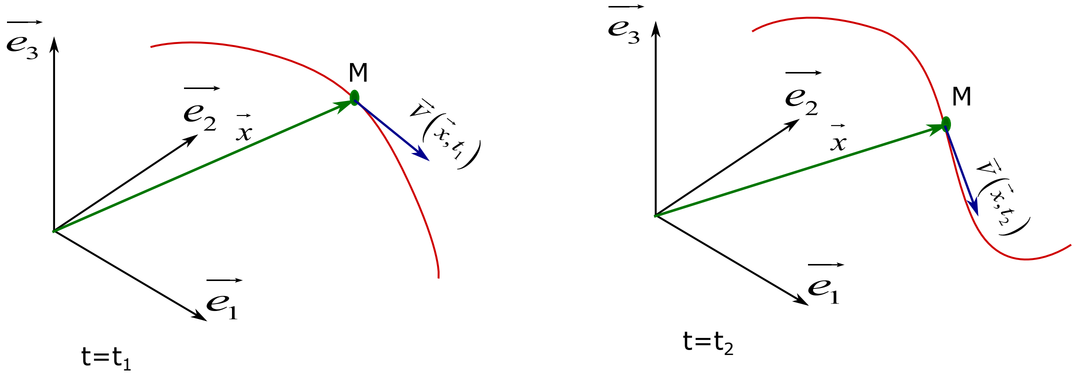


Figure 3.8: Eulerian description.

We thus again obtain a vector-valued function of 4 scalar variables, but unlike (3.2), the spatial variables x_1, x_2, x_3 are related to the **current configuration** and no longer to a reference configuration: they therefore no longer identify particles over time.

Any physical quantity is similarly defined on κ_t in the form

$$\forall t, \forall M \in \Omega_t, \quad B = b\left(\vec{x},t\right) \tag{3.13}$$

As a general rule, to make formulas easier to interpret at a glance, lowercase letters (such as $\vec{x}$) are used for functions related to the Eulerian description, and uppercase letters (such as $\vec{X}$) for functions related to the Lagrangian description.

3.5.2 Determination of Trajectories

It is clear that the Eulerian description of the evolution of a system S is obtained immediately once its Lagrangian description is known. Indeed, by equating the two expressions of the velocity of particle $\vec{X}$ in R at time t, and then the two expressions of the quantity B for particle $\vec{X}$ at time t, one obtains:

$$\begin{cases} \vec{v_t}\left(x,t\right) = \vec{v}\left(\vec{X},t\right) = \vec{v}\left(\vec{\psi}\left(\vec{x},t\right),t\right) \\ b\left(x,t\right) = B\left(\vec{X},t\right) = B\left(\vec{\psi}\left(\vec{x},t\right),t\right) \end{cases} \tag{3.14}$$

Conversely, it must be verified that the Eulerian description, introduced intuitively in the previous subsection, is indeed equivalent to the Lagrangian description. For this, it is enough to verify that (3.12) actually makes it possible to reconstruct the function $\vec{\phi}$ in Formula (3.2), that is, one must solve the problem of determining the vector-valued function $\vec{\phi}$ satisfying

$$\begin{cases} \dfrac{\partial \vec{\phi}\left(\vec{X},t\right)}{\partial t} = \vec{v_t}\left(\vec{\phi}\left(\vec{X},t\right),t\right) \\ \vec{\phi}\left(\vec{X},0\right) = \vec{X}, \quad \text{initial condition} \end{cases} \tag{3.15}$$

This amounts to determining the trajectory of every particle together with its time law. This problem can also be written in differential form:

$$\begin{cases} \overrightarrow{dx} = \overrightarrow{v_t}\left(\overrightarrow{x},t\right)dt \\ \overrightarrow{x}_{|t=0} = \overrightarrow{X}, \quad \text{initial condition} \end{cases} \tag{3.16}$$

which represents a system of 3 differential equations for the 3 unknown scalar functions x_1, x_2, x_3 of the variable t.

Subject to regularity conditions on the function $\overrightarrow{v_t}$, one can determine the unique solution for each initial condition, which then takes the desired form:

$$\overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{X},t\right)$$

thus completing the determination of the trajectories.

The set of trajectories in R of all particles constitutes a family of curves with 3 parameters (the coordinates X_i of M_0).

3.5.3 Streamlines

For certain applications, it is useful to introduce a third family of **geometric curves** arising from the definition of the motion of a continuum medium. At a given instant T, the streamlines of the motion in the reference frame R are defined as the envelope curves of the velocity field vectors $\overrightarrow{v_t}\left(\overrightarrow{x},T\right)$.

These lines are therefore defined in R by the differential system

$$\frac{dx_1}{v_1\left(\overrightarrow{x},T\right)} = \frac{dx_2}{v_2\left(\overrightarrow{x},T\right)} = \frac{dx_3}{v_3\left(\overrightarrow{x},T\right)} \tag{3.17}$$

This is a system of 2 differential equations in x_1, x_2, x_3. The streamlines in R at time T constitute **a family of geometric curves with** 2 **parameters**.

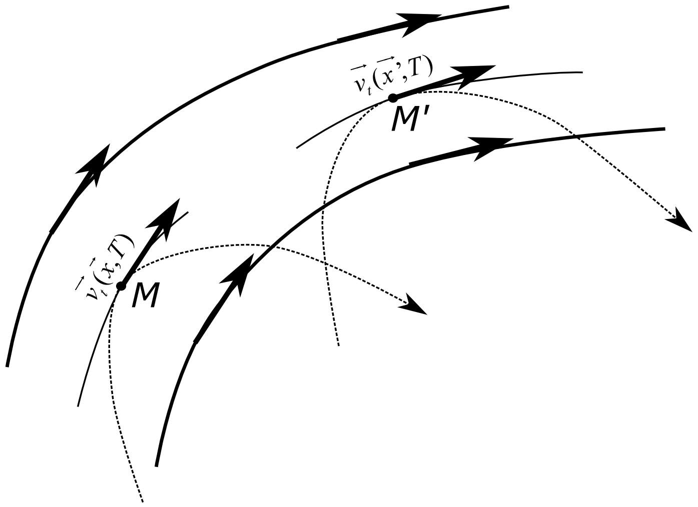


Figure 3.9: Streamlines at time T (dashed lines: trajectories of two particles)

An example of streamline visualization is shown in Figure 3.10.

Figure 3.10: Laser-sheet visualization of the streamlines of the transverse flow around the Concorde. (ONERA image.)

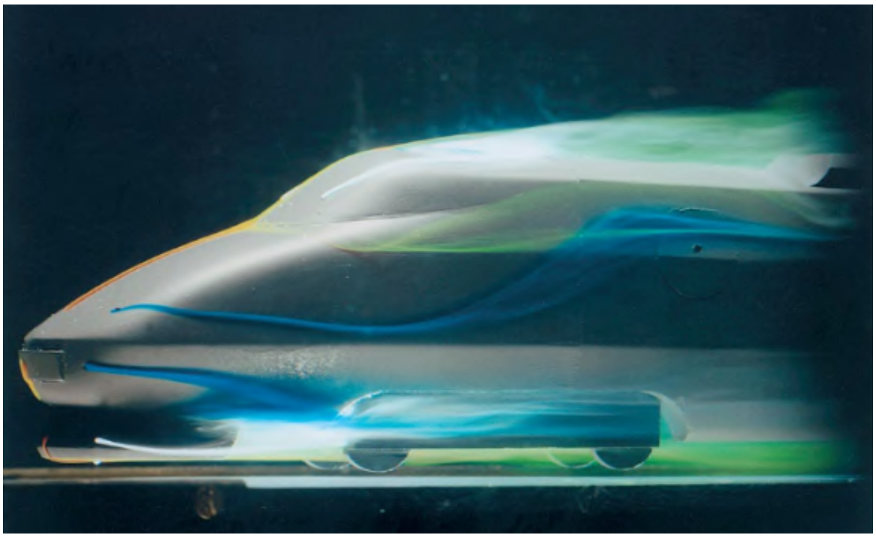


Figure 3.11: Flow around a high-speed train nose model. (ONERA image.)

3.5.4 Steady Motions

The motion is said to be steady in a reference frame R if, in its Eulerian description, $\vec{v_t}\,(\vec{x},t)$ is independent of t and depends only on the coordinates of the geometric point M. The following properties then result.

- In determining the trajectories and their time laws through the differential system (3.16), the problem decouples into a purely geometric problem and a time-scheduling problem.

Indeed, since

$$\overrightarrow{v_t}\left(\overrightarrow{x},t\right) \equiv \overrightarrow{v}\left(\overrightarrow{x}\right) \tag{3.18}$$

one obtains from (3.16) the system

$$\frac{dx_1}{v_1\left(\overrightarrow{x}\right)} = \frac{dx_2}{v_2\left(\overrightarrow{x}\right)} = \frac{dx_3}{v_3\left(\overrightarrow{x}\right)} \tag{3.19}$$

which is identical to the system that, at each instant T, determines the streamlines (3.17), itself independent of T.

Thus, the trajectories then form a family of geometric curves with 2 *parameters, identical to the family of streamlines, which in this case become time-independent.*

3.6 Exercise

3.6.1 Statement

Consider the flow defined by:

$$\overrightarrow{V} = \begin{cases} V_1 = 2.t - 5 \\ V_2 = t - 1 \end{cases}$$

1. Determine, as functions of the parameter t, the trajectories of three material points A, B, and C passing through the origin at times $t = 0$, $t = 1$, and $t = 3$.
2. Determine, in parametric form with respect to t, the equations of a streakline associated with a point of coordinates (y_1, y_2) at time T.
3. Plot the streakline associated with the origin at time $T = 3$.
4. Show that the streamlines at time τ are parallel straight lines.

3.6.2 Solution

- Determine, as functions of the parameter t, the trajectories of three material points A, B, and C passing through the origin at times $t = 0$, $t = 1$, and $t = 3$.

$$\frac{dx_1}{dt} = 2.t - 5$$
$$dx_1 = (2.t - 5).dt$$
$$x_1 = t^2 - 5.t + X_1$$

$$\frac{dx_2}{dt} = t - 1$$
$$dx_2 = (t - 1).dt$$
$$x_2 = \frac{t^2}{2} - t + X_2$$

$$\begin{cases} x_1 = t^2 - 5.t + X_1 \\ x_2 = \frac{t^2}{2} - t + X_2 \end{cases}$$

Particle A passes through the origin at time $t = 0s$ $(x_1 = 0, x_2 = 0)$

$$\begin{cases} 0 = X_1 \\ 0 = X_2 \end{cases}$$

Particle B passes through the origin at time $t = 1s$ $(x_1 = 0, x_2 = 0)$

$$\begin{cases} 0 = 1^2 - 5.1 + X_1 \\ 0 = \frac{1^2}{2} - 1 + X_2 \end{cases}$$

$$\begin{cases} 4 = X_1 \\ \frac{1}{2} = X_2 \end{cases}$$

Particle C passes through the origin at time $t = 3s$ $(x_1 = 0, x_2 = 0)$

$$\begin{cases} 0 = 3^2 - 5.3 + X_1 \\ 0 = \frac{3^2}{2} - 3 + X_2 \end{cases}$$

$$\begin{cases} 6 = X_1 \\ -\frac{3}{2} = X_2 \end{cases}$$

The trajectory of particle A is:

$$\begin{cases} x_1 = t^2 - 5.t \\ x_2 = \frac{t^2}{2} - t \end{cases}$$

The trajectory of particle B is:

$$\begin{cases} x_1 = t^2 - 5.t + 4 \\ x_2 = \frac{t^2}{2} - t + \frac{1}{2} \end{cases}$$

The trajectory of particle C is:

$$\begin{cases} x_1 = t^2 - 5.t + 6 \\ x_2 = \frac{t^2}{2} - t - \frac{3}{2} \end{cases}$$

- Determine, in parametric form with respect to t, the equations of a streakline associated with a point of coordinates (y_1, y_2) at time T.

The position of the particles at time T is of the form:

$$\begin{cases} x_1 = T^2 - 5.T + X_1 \\ x_2 = \frac{T^2}{2} - T + X_2 \end{cases}$$

The equations of the particles passing through the point of coordinates (y_1, y_2) at time $t < T$ are

$$\begin{cases} y_1 = t^2 - 5.t + X_1 \\ y_2 = \frac{t^2}{2} - t + X_2 \end{cases}$$

From these equations, one determines the initial positions of the particles passing through the point of coordinates (y_1, y_2):

$$\begin{cases} X_1 = y_1 - t^2 + 5.t \\ X_2 = y_2 - \frac{t^2}{2} + t \end{cases}$$

Substituting into the first equation gives the equation of the streakline:

$$\begin{cases} x_1 = -t^2 + 5.t + y_1 + T^2 - 5.T \\ x_2 = -\frac{t^2}{2} + t + y_2 + \frac{T^2}{2} - T \end{cases}$$

- Plot the streakline associated with the origin at time $T = 3$.

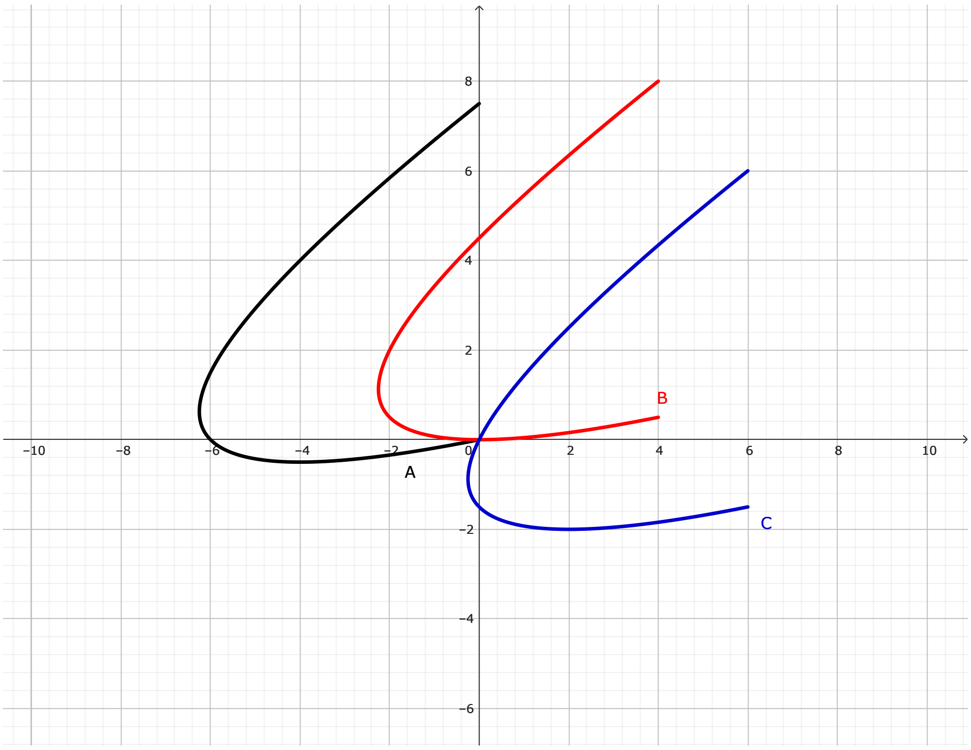


Figure 3.12: Trajectories of the three points A, B, and C for $t = 0..5s$

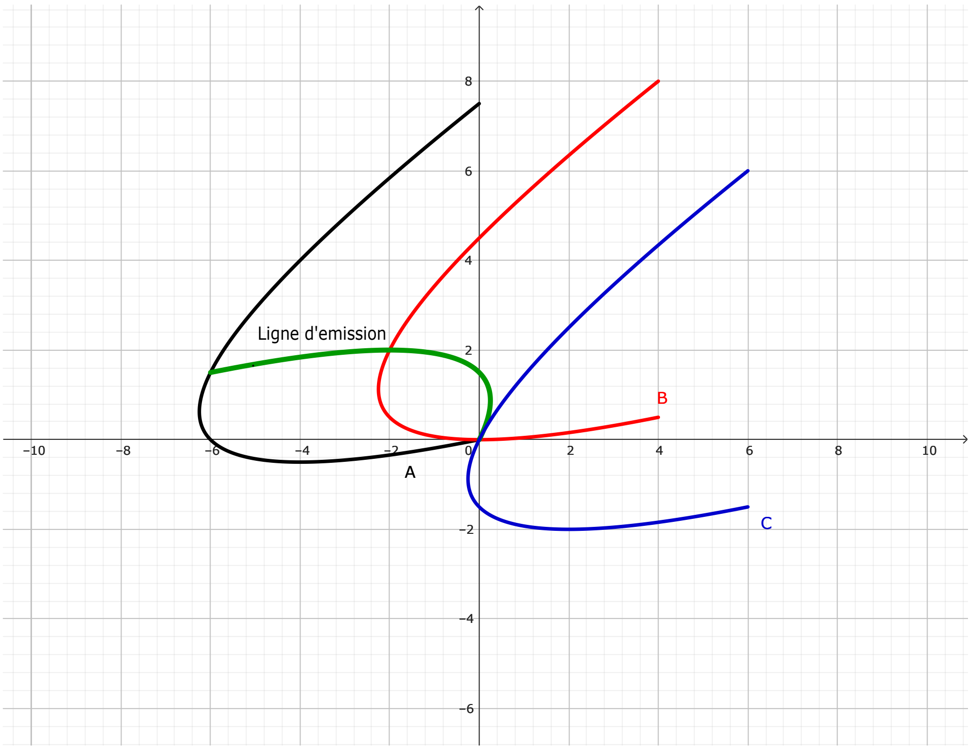


Figure 3.13: Streakline associated with the origin at time $T = 3$

The equation of the streakline associated with the origin at time $T = 3$ is

$$\begin{cases} x_1 = -t^2 + 5.t - 6 \\ x_2 = -\frac{t^2}{2} + t + \frac{3}{2} \end{cases}$$

- Show that the streamlines at time τ are parallel straight lines.

The equation of the streamlines is of the form:

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$$

$$\frac{dx_1}{2.\tau - 5} = \frac{dx_2}{\tau - 1}$$

$$dx_2 = \frac{\tau - 1}{2.\tau - 5}.dx_1$$

$$x_2 = \frac{\tau - 1}{2.\tau - 5}.x_1 + C_0$$

which represents parallel straight lines.

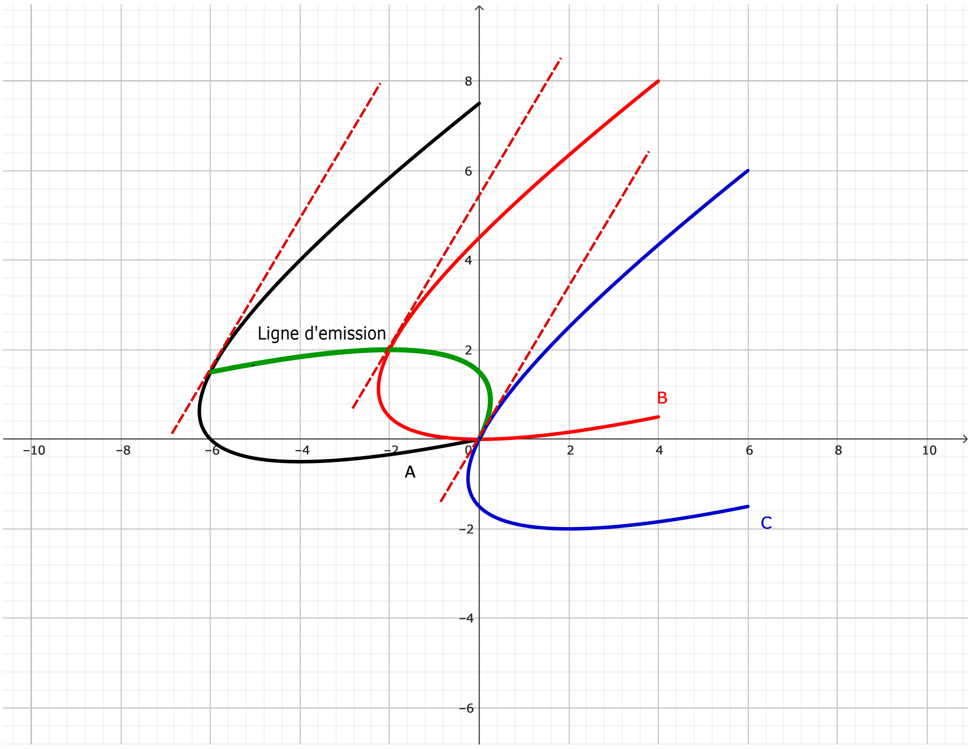
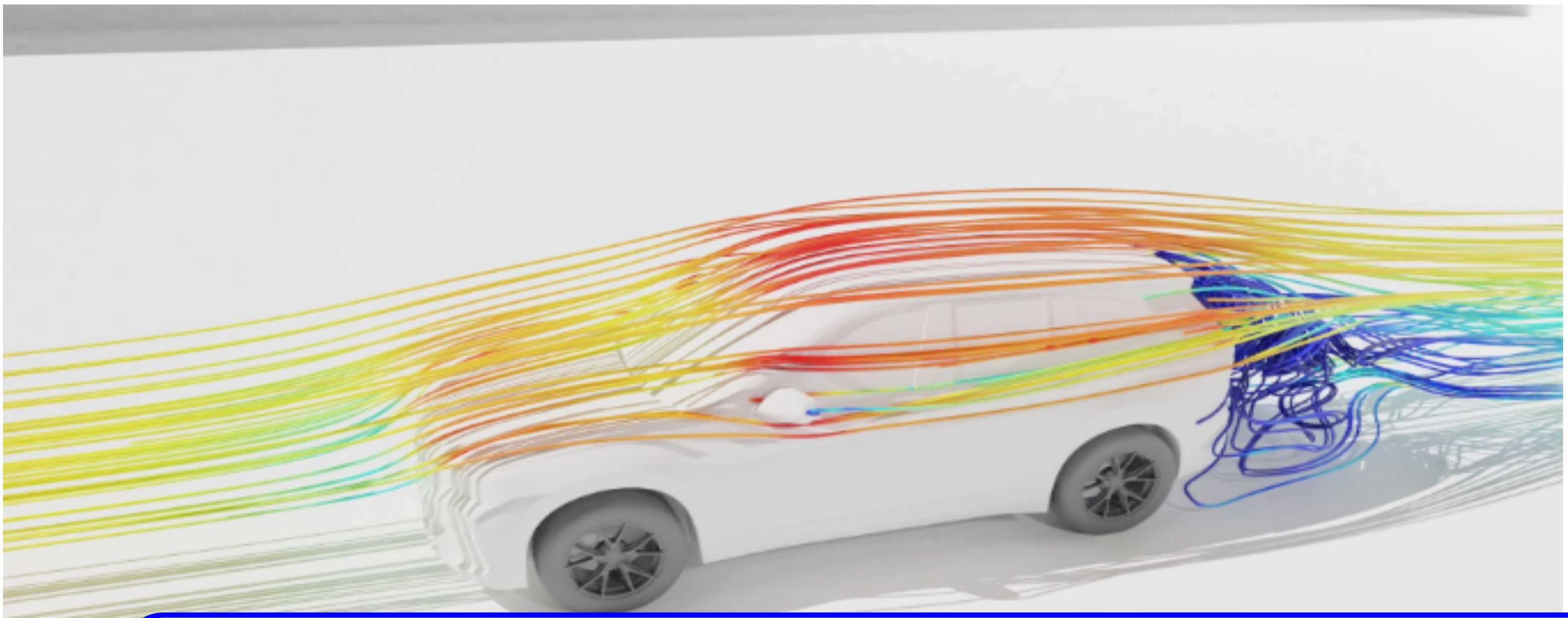


Figure 3.14: Streamlines at time $\tau = 3$



4. Study of the Deformation of a Continuum Medium

4.1 Introduction

Figure 4.1 shows the indentation of a solid by a cylinder. We have chosen the undeformed configuration as the initial configuration. A grid of lines is shown on both configurations. After deformation (in configuration κ_t), the grid of lines, initially orthogonal, becomes distorted.

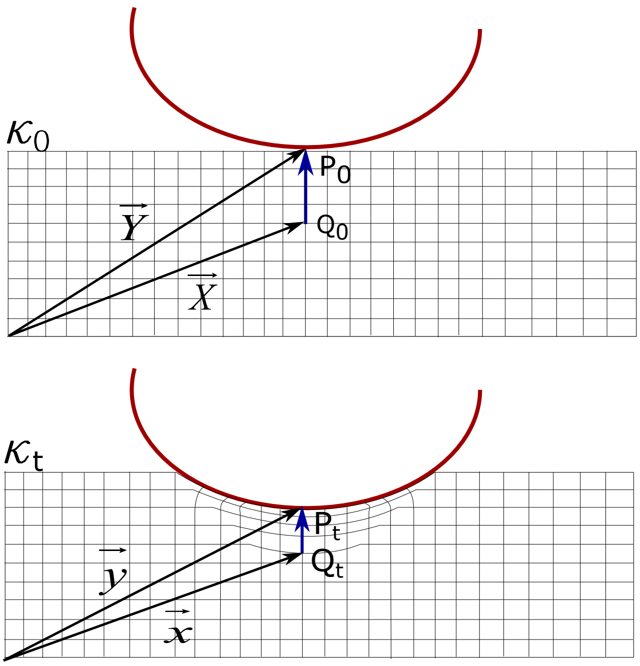


Figure 4.1: Grid of lines drawn on a solid before indentation and after indentation by a cylinder.

We observe that the distortion of the grid:

- is not constant throughout the solid;
- results in a change of the angles that were initially right angles;

- results in a change of lengths.

In the remainder of this chapter, we introduce a measure of angular distortion and a measure of length variation. These measures must be local, since the distortions of the grid may vary throughout the solid.

4.2 Gradient of a Transformation

4.2.1 Transformation Gradient Tensor

Definition

Let R be a reference frame in which a frame has been chosen, consisting of an origin O and an orthonormal Cartesian basis. Let C be any part of a material system. Let κ_0 denote the configuration of C at the initial instant and κ_t its configuration at time t. Consider two material points Q and P occupying, at the initial instant, the positions Q_0, P_0, and at time t the positions Q_t, P_t. In the initial configuration, Q_0 and P_0 are described by two vectors

$$\overrightarrow{OP_0} = \overrightarrow{X} = X_P \overrightarrow{e_P} \quad \text{and} \quad \overrightarrow{OQ_0} = \overrightarrow{Y} = Y_P \overrightarrow{e_P}$$

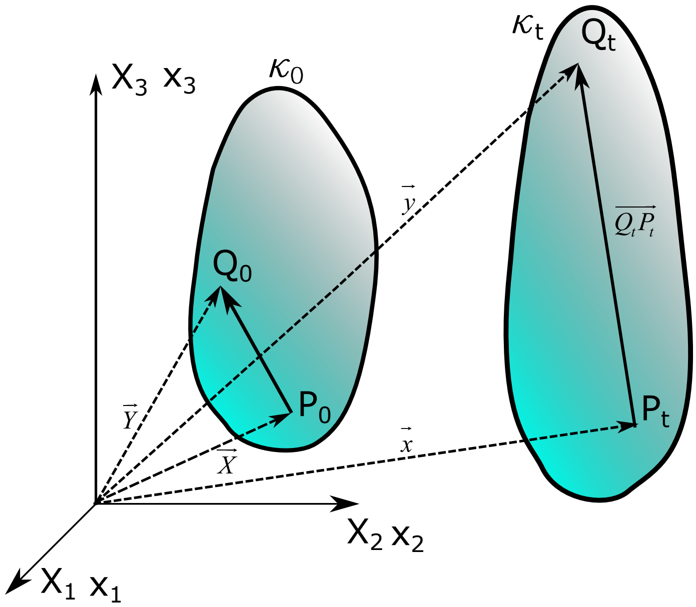


Figure 4.2: Initial and current configurations of a deformed solid.

One of the objectives stated in the introduction is to determine whether the distance between points P_t and Q_t changes during the transformation. To estimate the distance between P_t and Q_t, we express the vector $\overrightarrow{P_tQ_t}$ as a function of the vector.

The transition from the initial configuration to the configuration at time t can be described by the transformation. The vector $\overrightarrow{P_tQ_t}$ can therefore be written as

$$\overrightarrow{P_tQ_t} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{y} - \overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{Y}, t\right) - \overrightarrow{\phi}\left(\overrightarrow{X}, t\right)$$

$$y_i - x_i = \phi_i\left(Y_1, Y_2, Y_3\right) - \phi_i\left(X_1, X_2, X_3\right) \qquad (i = 1, 2, 3)$$

The vector $\overrightarrow{P_tQ_t}$ is the increment of the function ϕ for an increment of the variable. Its first-order Taylor expansion is given by

$$y_i - x_i = \frac{\partial \phi_i}{\partial X_P}(X_1,X_2,X_3)(Y_P - X_P) +$$
$$\left[(Y_1-X_1)^2+(Y_2-X_2)^2+(Y_3-X_3)^2\right]^{1/2}\alpha_i(Y_1-X_1,Y_2-X_2,Y_3-X_3,t)$$

where α_i is a function that tends to 0 when Y_i tends to X_i.

In vector form, one may write:

$$\boxed{\begin{array}{c}\vec{y}-\vec{x}=\overline{\overline{F}}\left(\vec{X},t\right)\left(\vec{Y}-\vec{X}\right)+\vec{\alpha}\left(\vec{Y}-\vec{X},t\right)\left\|\vec{Y}-\vec{X}\right\| \\ \overrightarrow{P_tQ_t}=\overline{\overline{F}}\left(\overrightarrow{OP_0},t\right)\overrightarrow{P_0Q_0}+\vec{\alpha}\left(\overrightarrow{P_0Q_0},t\right)\left\|\overrightarrow{P_0Q_0}\right\|\end{array}} \tag{4.1}$$

The tensor $\overline{\overline{F}}\left(\vec{X},t\right)=\overrightarrow{grad}\left[\vec{\phi}\left(\vec{X},t\right)\right]$ is called the **transformation gradient tensor**. $\overline{\overline{F}}$ depends on the point considered and on time t. The components of $\overline{\overline{F}}$ are given by

$$\boxed{\begin{array}{c}\overline{\overline{F}}\left(\vec{X},t\right)=\overrightarrow{grad}\left[\vec{\phi}\left(\vec{X},t\right)\right]=\overrightarrow{grad}\left[\vec{x}\left(\vec{X},t\right)\right] \\ F_{ij}=\frac{\partial x^i}{\partial X^j}\end{array}} \tag{4.2}$$

Karl Jacobi
1804 - 1851

R Charles Gustave Jacob Jacobi, born in 1804, was a German mathematician best known for his work on elliptic integrals, partial differential equations, and their applications to analytical mechanics. He was the brother of the physicist Moritz von Jacobi, the discoverer of electroplating.

Example

Figure 4.3 shows a particular transformation. The whole solid is first translated by 4 units in the x_1 direction. Then a transformation called simple shear is applied.

The transformation can be written as follows:

$$\left\{\begin{array}{c} x_1 = X_1 + 4 \\ x_2 = 3.\frac{X_2}{2} \\ x_3 = X_3 \end{array}\right.$$

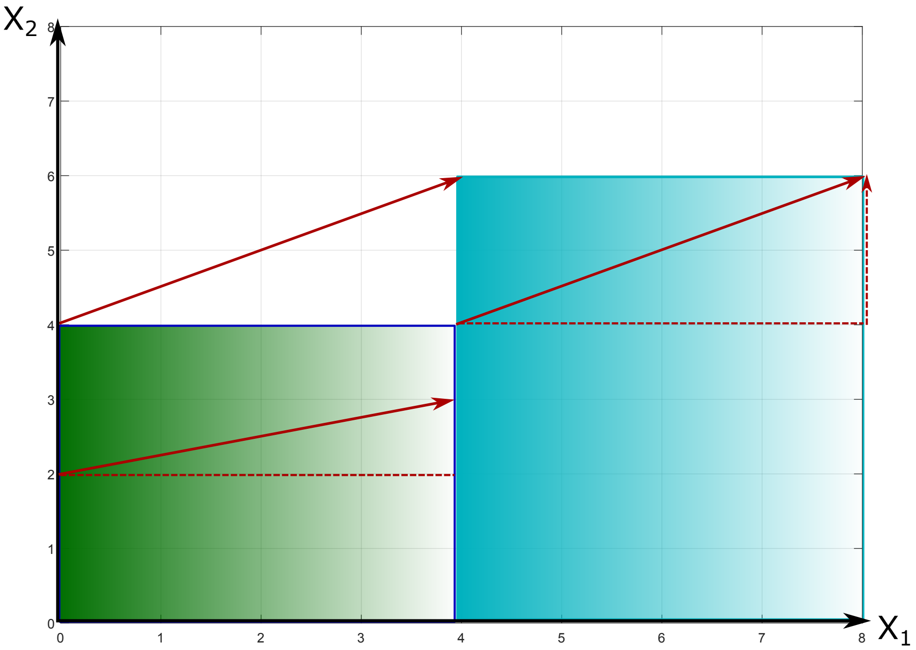


Figure 4.3: Example of a particular transformation: simple shear with translation.

The transformation gradient tensor is

$$\overline{\overline{F}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation of 4 units in the positive x_1 direction does not appear in the transformation gradient tensor. In the following, the overall motion of a solid will be called a rigid motion or rigid-body mode. The gradient tensor of this transformation does not depend on the coordinates of the initial configuration. This type of transformation is called a homogeneous transformation.

Homogeneous Transformations

If the tensor $\overline{\overline{F}}$ does not depend on the material point P_0, the transformation is said to be homogeneous.

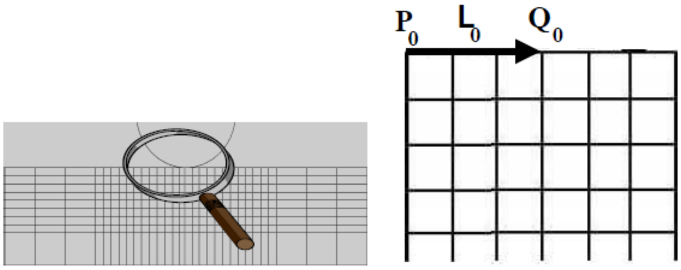
$$\boxed{\overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{X},t\right) = \overline{\overline{F}}(t)\,\overrightarrow{X} + \overrightarrow{B}(t)} \quad (4.3)$$

Geometrical Interpretation of the Transformation Gradient Tensor

Figures 4.4a, 4.4b, 4.5a, and 4.5b show the indentation of a solid by a rigid cylinder. Figures 4.4a and 4.4b correspond to the undeformed configuration. This configuration will be adopted as the initial configuration C_0. Figures 4.5a and 4.5b correspond to the deformed configuration, denoted C_t.

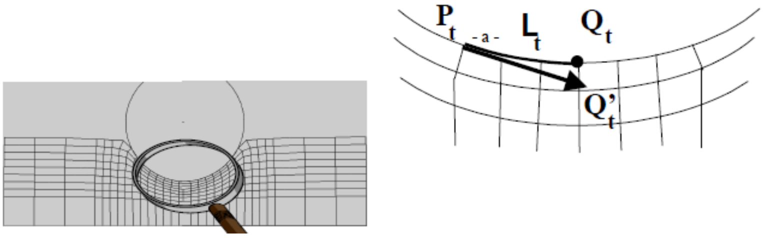
The line segment $L_0 = P_0Q_0$ in the initial configuration is transformed into a curve segment $L_t = P_tQ_t$

$$\overrightarrow{P_tQ_t} = \overline{\overline{F}}\left(\overrightarrow{OP_0},t\right)\overrightarrow{P_0Q_0} + \overrightarrow{\alpha}\left(\overrightarrow{P_0Q_0},t\right)\left\|\overrightarrow{P_0Q_0}\right\|$$



(a) (b)

Figure 4.4: Indentation of a solid by an infinite cylinder, with the undeformed configuration taken as reference



(a) (b)

Figure 4.5: Indentation of a solid by an infinite cylinder, with the deformed configuration shown

Let us set

$$\overrightarrow{P_tQ_t} = \overline{\overline{F}}\left(\overrightarrow{OP_0},t\right)\overrightarrow{P_0Q_0} + \overrightarrow{\alpha}\,(\overrightarrow{P_0Q_0},t)\left\|\overrightarrow{P_0Q_0}\right\|$$

We may write

$$\overrightarrow{P_tQ_t} = \overrightarrow{P_tQ_t}' + \overrightarrow{\alpha}\,(\overrightarrow{P_0Q_0},t)\left\|\overrightarrow{P_0Q_0}\right\|$$

Differential Notation

Equation 4.1 is exact whatever the length of the vector $\overrightarrow{P_0Q_0}$. However, the vector $\overrightarrow{P_tQ'_t} = \overline{\overline{F}}\left(\overrightarrow{OP_0},t\right)\overrightarrow{P_0Q_0}$ is a good approximation of the vector $\overrightarrow{P_tQ_t}$ only if Q_0 is close to P_0. In the following, we consider the points P_0 and Q_0 to be infinitely close, so that $\overrightarrow{P_0Q_0} = d\overrightarrow{X}$, $\overrightarrow{P_tQ'_t} = d\overrightarrow{x}$.

If one sets (Figure 4.6) $\overrightarrow{P_tQ_t} = \overrightarrow{OQ_t} - \overrightarrow{OP_t} = \overrightarrow{y} - \overrightarrow{x} = \Delta\overrightarrow{x}$ and $\overrightarrow{P_0Q_0} = \overrightarrow{OQ_0} - \overrightarrow{OP_0} = \overrightarrow{Y} - \overrightarrow{X} = d\overrightarrow{X}$, Equation 4.3 becomes

$$\boxed{\Delta\overrightarrow{x} = d\overrightarrow{x} + \overrightarrow{\alpha}\,(d\overrightarrow{X})\left\|d\overrightarrow{X}\right\|} \qquad (4.4)$$

Thus, identifying the vector $\overrightarrow{P_tQ_t}$ with $\overrightarrow{P_tQ'_t}$ amounts to approximating the increment $\Delta\overrightarrow{x}$ of the function $\overrightarrow{\phi}\,(\overrightarrow{X},t)$ by the vector $d\overrightarrow{x}$.

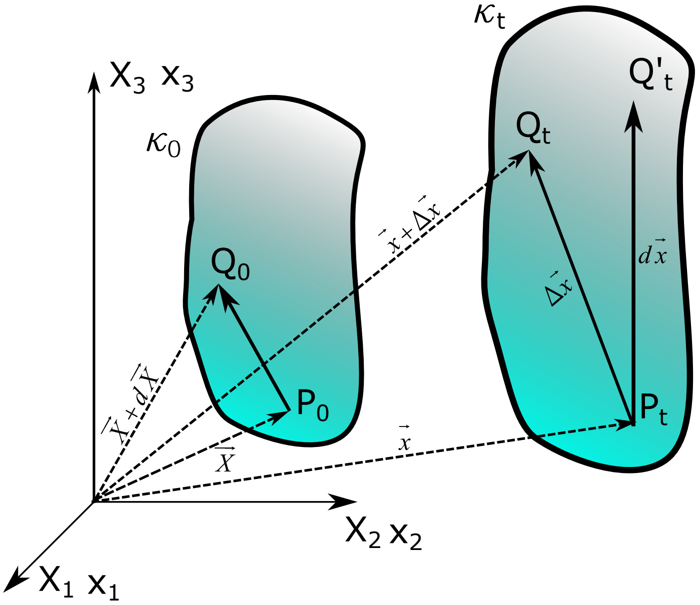


Figure 4.6: Transition from the representation of finite quantities to differential quantities.

4.2.2 Convective Transport of a Vector

Arc Element, Surface Element, Volume Element

In the reference configuration κ_0, let L_0 be a curve passing through P_0. In the vicinity of the point, this curve can be approximated by the tangent vector $d\vec{X}$. The vector $d\vec{X}$ will be called an elementary vector. An elementary surface element dS_0 is a parallelogram constructed from two elementary vectors $d\vec{X}$ and $d\vec{X'}$ issued from P_0. An elementary volume element dV_0 is a parallelepiped constructed from three elementary vectors $d\vec{X}$, $d\vec{X'}$, and $d\vec{X''}$. In the current configuration κ_t, the image L_t of the arc L_0 is identified with the vector $d\vec{x}$, the transported image of the vector $d\vec{X}$. The transported image dS_t of the surface dS_0 is identified with the surface spanned by the vectors $d\vec{x}$ and $d\vec{x'}$, transported from $d\vec{X}$ and $d\vec{X'}$.

Convective Transport of an Elementary Vector

Consider a curve L_0 passing through the points P_0 and Q_0 (Figure 4.7) in the reference configuration. In the current configuration, the curve L_0 is transformed into the curve L_t passing through $\overrightarrow{P_tQ_t}$. This curve is approximated by the transported image of the vector $d\vec{X}$, namely $d\vec{x}$. Introduce the unit vectors $\vec{\tau_0}$ and $\vec{\tau}$ such that $d\vec{X} = dl_0\vec{\tau_0}$ and $d\vec{x} = dl\,\vec{\tau}$.
Since $d\vec{x} = \overline{\overline{F}}d\vec{X}$,

$$\boxed{\vec{\tau}\,dl = \overline{\overline{F}}\,\vec{\tau_0}\,dl_0} \tag{4.5}$$

In summary, locally, an elementary vector in κ_0 is transformed into an elementary vector in κ_t such that

$$\boxed{d\vec{x} = \overline{\overline{F}}d\vec{X}} \tag{4.6}$$

4.3 Stretch Tensor

Let us again consider the problem of the indentation of a solid by an infinite cylinder (Figure 4.8). We propose to compare lengths and angles in the reference configuration and in the current

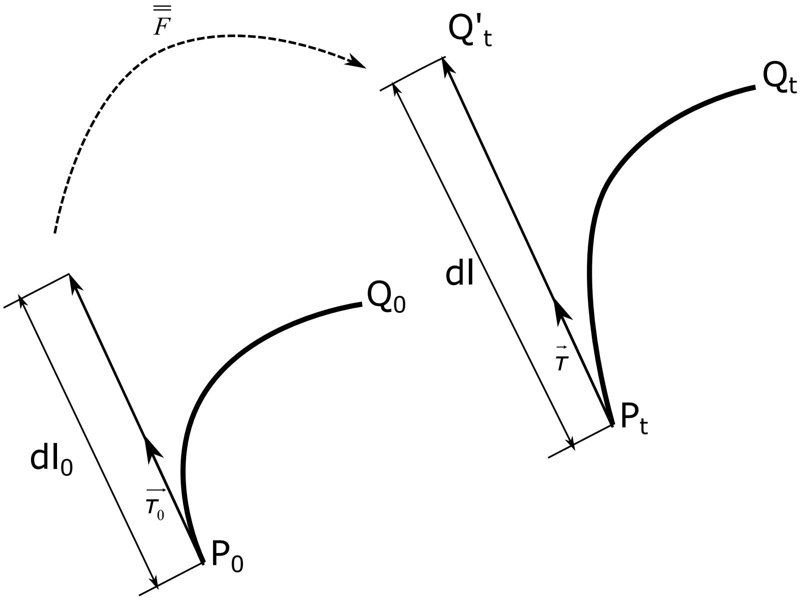


Figure 4.7: Transport of an elementary vector.

configuration. The variation of length depends on the orientation of the segment considered in the reference configuration. The most suitable measure for our problem therefore seems to be the scalar product of two elementary vectors. Indeed, the scalar product accounts for the relative orientation of vectors. The scalar product of a vector with itself allows lengths to be estimated. The scalar product of two different vectors allows the angle formed by these vectors to be estimated.

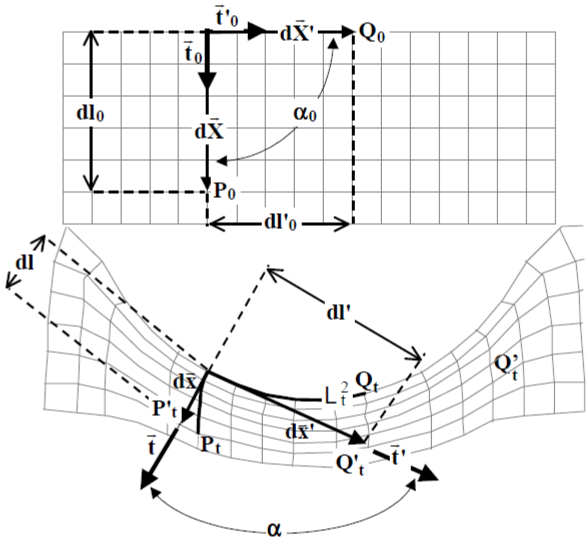


Figure 4.8: Indentation of a solid by an infinite cylinder. Reference configuration and deformed configuration.

4.3.1 Stretch Tensor

Definition

Let us compute the scalar product: $d\vec{x}^T.d\vec{x'} = \left(\overline{\overline{F}}d\vec{X}\right)^T\overline{\overline{F}}d\vec{X'}$

$$d\overrightarrow{x}^T.d\overrightarrow{x'} = d\overrightarrow{X}^T\left[\overline{\overline{F}}^T\overline{\overline{F}}\right]d\overrightarrow{X'}$$

$$d\overrightarrow{x}^T.d\overrightarrow{x'} = d\overrightarrow{X}^T\left[\overline{\overline{F}}^T\overline{\overline{F}}\right]d\overrightarrow{X'} = d\overrightarrow{X}^T\overline{\overline{C}}d\overrightarrow{X'}$$

The Cauchy-Green stretch tensor is defined by

$$\boxed{\overline{\overline{C}} = \overline{\overline{F}}^T\overline{\overline{F}}} \tag{4.7}$$

Louis Augustin Cauchy
1789 - 1857

R Augustin-Louis, Baron Cauchy, born in Paris in 1789 and deceased in Sceaux in 1857, was a French mathematician and a member of the École Polytechnique. He was one of the most prolific mathematicians of all time, with nearly 800 publications and seven books.

The tensor $\overline{\overline{C}} = \overline{\overline{F}}^T\overline{\overline{F}}$ is called the Cauchy-Green stretch tensor. This tensor depends on the initial configuration and makes it possible to compute the scalar product of two vectors in the current configuration as a function of the scalar product of the corresponding vectors in the initial configuration. Since, as a bilinear form, it acts on two vectors related to the initial configuration, $\overline{\overline{C}}$ is said to be a Lagrangian tensor. The components of the tensor $\overline{\overline{C}}$ are dimensionless.

Properties of the Cauchy-Green Stretch Tensor

$\overline{\overline{C}}$ is symmetric (i.e. equal to its transpose). Indeed,

$$\overline{\overline{C}}^T = \left(\overline{\overline{F}}^T\overline{\overline{F}}\right)^T = \overline{\overline{F}}\left(\overline{\overline{F}}^T\right)^T = \overline{\overline{F}}^T\overline{\overline{F}} = \overline{\overline{C}}$$

The bilinear form $\overrightarrow{X}^T\overline{\overline{C}}\overrightarrow{X}$ is positive definite. Indeed,

$$\det\left[\overline{\overline{C}}\right] = \det\left[\overline{\overline{F}}^T\overline{\overline{F}}\right] = \det\left(\overline{\overline{F}}\right)^2 = J^2 > 0 \quad \text{and} \quad d\overrightarrow{X}^T\overline{\overline{C}}d\overrightarrow{X} = \|d\overrightarrow{x}\|^2 > 0$$

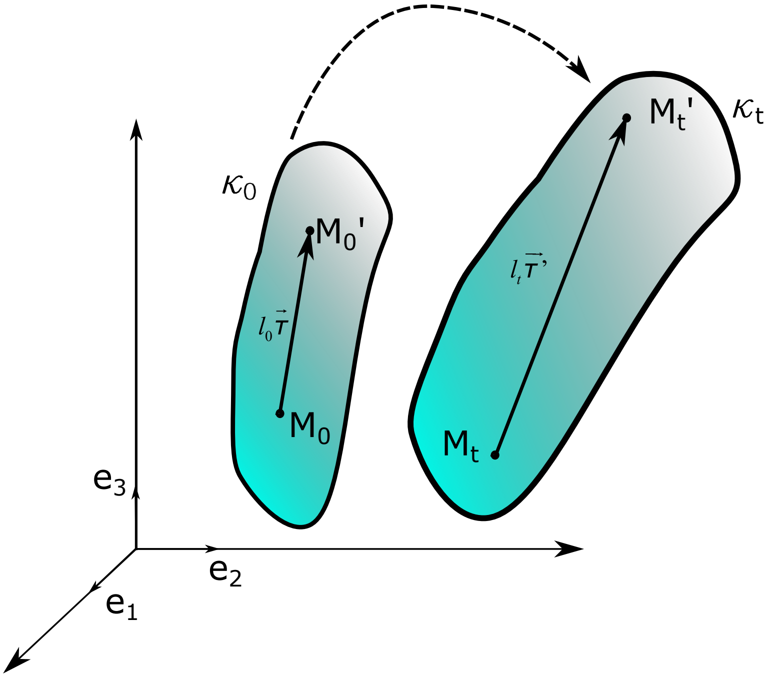


Figure 4.9: Evolution of lengths.

4.3.2 Stretch in a Given Direction

Let $\overrightarrow{M_0M_0'} \in \kappa_0$ be an elementary vector of length l_0, and let $\overrightarrow{M_tM_t'}$ of length l_t be its transformed image in κ_t. The vector $\overrightarrow{M_0M_0'}$ has direction given by the unit vector $\overrightarrow{\tau}$, see Figure 4.9.

$$\overrightarrow{M_0M_0{}'} = l_0\overrightarrow{\tau}$$

Let us compute the length l_t of $\overrightarrow{M_tM_t'}$ as a function of that of $\overrightarrow{M_0M_0'}$. According to Equation 4.6, we have:

$$\overrightarrow{M_tM_t{}'} = \overline{\overline{F}}\,\overrightarrow{M_0M_0{}'}$$

We have:

$$l_t = \left|\overrightarrow{M_tM_t{}'}\right| = \sqrt{\overrightarrow{M_tM_t{}'}.\overrightarrow{M_tM_t{}'}}$$

$$l_t = \sqrt{\overline{\overline{F}}\,\overrightarrow{M_0M_0{}'}.\overline{\overline{F}}\,\overrightarrow{M_0M_0{}'}} = \sqrt{\overrightarrow{M_0M_0{}'}\,\overline{\overline{F}}^T.\overline{\overline{F}}\,\overrightarrow{M_0M_0{}'}}$$

$$l_t = \sqrt{l_0\overrightarrow{\tau}\,\overline{\overline{F}}^T.\overline{\overline{F}}\,l_0\overrightarrow{\tau}}$$

$$l_t = l_0\sqrt{\overrightarrow{\tau}\,\overline{\overline{F}}^T.\overline{\overline{F}}\,\overrightarrow{\tau}}$$

Let $\lambda(\tau)$ denote the unit stretch in the direction $\overrightarrow{\tau}$:

$$\lambda(\tau) = \frac{l_t - l_0}{l_0}$$

$$\lambda\left(\tau\right)=\frac{l_0\sqrt{\overrightarrow{\tau}\overline{\overline{F}}^T.\overline{\overline{F}}\overrightarrow{\tau}}-l_0}{l_0}$$

$$\lambda\left(\tau\right)=\sqrt{\overrightarrow{\tau}\overline{\overline{F}}^T.\overline{\overline{F}}\overrightarrow{\tau}}-1$$

The unit stretch in the direction $\overrightarrow{\tau}$ is therefore given in terms of the Cauchy-Green stretch tensor $\overline{\overline{C}}$ by

$$\boxed{\lambda\left(\tau\right)=\sqrt{\overrightarrow{\tau}\overline{\overline{C}}\overrightarrow{\tau}}-1} \tag{4.8}$$

Finally, the length of the vector $\overrightarrow{M_tM_t'}$ is given by

$$\boxed{l_t=l_0\sqrt{\overrightarrow{\tau}\overline{\overline{C}}\overrightarrow{\tau}}} \tag{4.9}$$

4.3.3 Shear Between Two Orthogonal Directions

A change of metric induces changes in angles. We are particularly interested in the change of angle between two vectors initially orthogonal, see Figure 4.10.

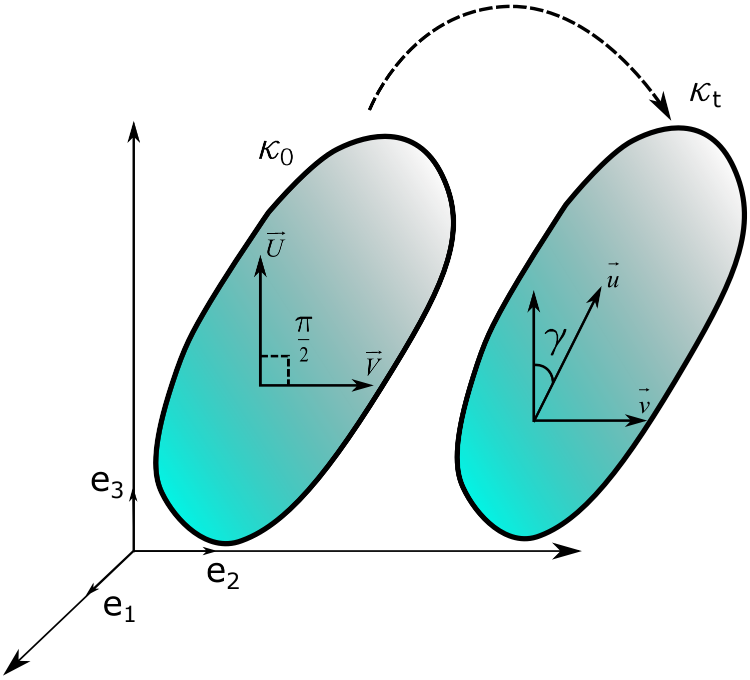


Figure 4.10: Evolution of angles.

Let $\overrightarrow{U}$ and $\overrightarrow{V}$ be two vectors initially orthogonal in κ_0:

$$\overrightarrow{U}=l_0\overrightarrow{\tau}$$

$$\overrightarrow{V}=l_0{}'\overrightarrow{\tau'}$$

$$\left(\overrightarrow{\tau}, \overrightarrow{\tau'}\right) = \frac{\pi}{2}$$

The image of $\overrightarrow{U}$ in κ_t is the vector $\overrightarrow{u}$ of length l_t, and the image of $\overrightarrow{V}$ is the vector $\overrightarrow{v}$ of length $l_t{}'$ such that

$$\overrightarrow{u} = \overline{\overline{F}}\overrightarrow{U}$$
$$\overrightarrow{v} = \overline{\overline{F}}\overrightarrow{V}$$

The shear between the two orthogonal directions $\overrightarrow{\tau}$ and $\overrightarrow{\tau'}$ is defined as the angle γ such that

$$(\overrightarrow{u}, \overrightarrow{v}) = \frac{\pi}{2} - \gamma$$

It is easy to show that

$$\boxed{\sin(\gamma) = \frac{\overrightarrow{\tau'}\overline{\overline{C}}\overrightarrow{\tau}}{\sqrt{\overrightarrow{\tau}\overline{\overline{C}}\overrightarrow{\tau}}\sqrt{\overrightarrow{\tau'}\overline{\overline{C}}\overrightarrow{\tau'}}}} \tag{4.10}$$

4.3.4 Evolution of Volumes

Let M_0 be a point of the configuration κ_0. Consider an elementary parallelepiped with vertex M_0, constructed from the three elementary vectors $d\overrightarrow{X}_1$, $d\overrightarrow{X}_2$, and $d\overrightarrow{X}_3$, see Figure 4.11.

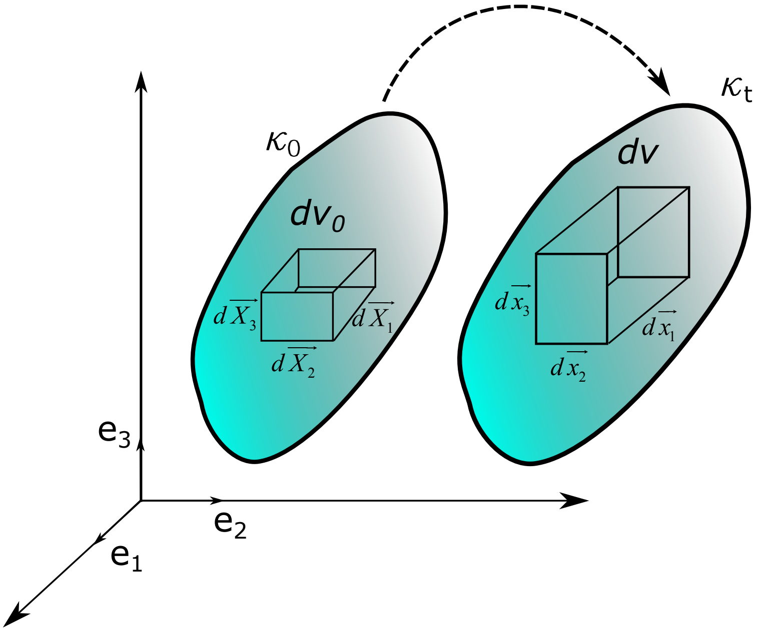


Figure 4.11: Evolution of volumes.

Its algebraic volume is given by

$$dv = d\overrightarrow{x_1} \cdot (d\overrightarrow{x_2} \wedge d\overrightarrow{x_3})$$

One has the following property:

$$\boxed{dv = \det(\overline{\overline{F}})dv_0} \tag{4.11}$$

with

$$\det(\overline{\overline{F}}) = \left|\frac{\partial x_i}{\partial X_j}\right|, \quad i,j = 1,2,3$$

We denote $\det(\overline{\overline{F}}) = J$, the Jacobian of the transformation. Finally, an elementary volume dv_0 belonging to κ_0 is transformed into dv in κ_t according to

$$\boxed{dv = J dv_0} \tag{4.12}$$

R $\overrightarrow{\phi}$ is bijective, bicontinuous, and continuously differentiable. It follows that $J\left(\overrightarrow{X},t\right)$ is continuous with respect to $\overrightarrow{X}$ and t. Moreover, $J\left(\overrightarrow{X},t\right)$ can be neither zero nor infinite because the Jacobian matrices of $\overrightarrow{\phi}$ and $\overrightarrow{\phi}^{-1}$ must be invertible. Thus, $J\left(\overrightarrow{X},t\right)$ preserves a constant sign throughout the transformation. Since

$$J\left(\overrightarrow{X},0\right) = \left|\frac{\partial x_i}{\partial X_j}\right| = 1 > 0, \quad i,j = 1,2,3$$

we have

$$\forall t,\ 0 < J < +\infty$$

4.3.5 Polar Decomposition

In this paragraph, we introduce a tool that can characterize the transformation gradient tensor $\overline{\overline{F}}$.

Principal Directions and Principal Stretches of the Cauchy Tensor

The Cauchy stretch tensor is symmetric positive definite; it is therefore diagonalizable. Its eigenvalues $(C_i; i = 1,2,3)$ are strictly positive; $C_i = \lambda_i^2; i = 1,2,3$, and its eigenvectors are pairwise orthogonal.

Let $\overrightarrow{U_i}; i = 1,2,3$ be its eigenvectors in κ_0. One then obtains

$$\boxed{\overrightarrow{U_i}\cdot\overrightarrow{U_j} = \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}} \tag{4.13}$$

Let $\overrightarrow{u_i}; i = 1,2,3$ be the convectively transported vectors in κ_t corresponding to $\overrightarrow{U_i}$ through $\overline{\overline{F}}$:

$$\overrightarrow{u_i} = \overline{\overline{F}}\,\overrightarrow{U_i}$$

Let us compute the scalar product:

$$\overrightarrow{u_i}\cdot\overrightarrow{u_j} = \overline{\overline{F}}\,\overrightarrow{U_i}\cdot\overline{\overline{F}}\,\overrightarrow{U_j}$$

$$= \overrightarrow{U_i}\cdot\overline{\overline{F}}^T\overline{\overline{F}}\cdot\overrightarrow{U_j}$$

$$= \overrightarrow{U_i}\cdot\overline{\overline{C}}\cdot\overrightarrow{U_j}$$

$$= \overrightarrow{U_i} \cdot \lambda_j^2 \cdot \overrightarrow{U_j}$$

$$= \lambda_j^2 \cdot \delta_{ij}$$

Hence the following statement: the principal directions of the Cauchy tensor $\overline{\overline{C}}$ in κ_0 are convectively transported into κ_t as orthogonal directions. During the transformation, the vectors of the principal basis of $\overline{\overline{C}}$ in κ_0 undergo a rotation and a stretch, see Figure 4.12.

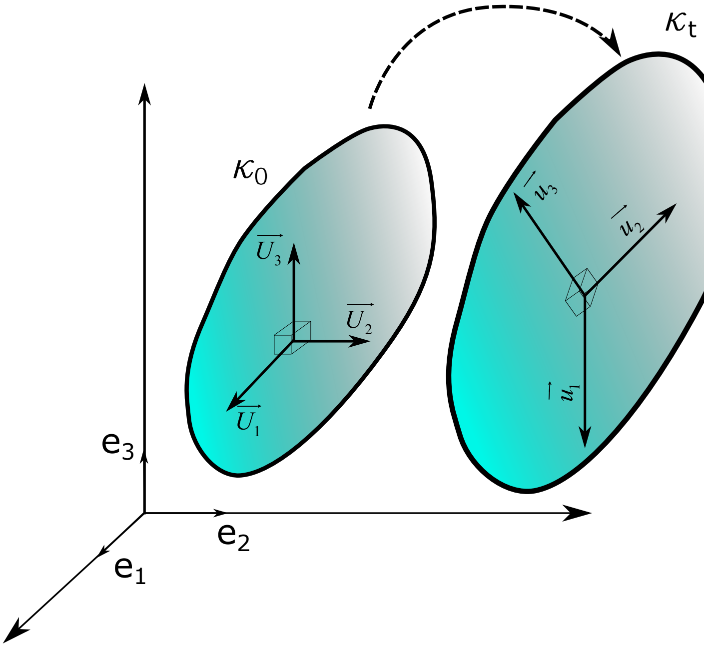


Figure 4.12: Transformation of the principal basis of the Cauchy stretch tensor.

Polar Decomposition

In order to characterize the tensor $\overline{\overline{F}}$, one can characterize the matrix associated with the corresponding linear mapping. For this, it is sufficient to compute the images of the vectors of a basis (one may choose the principal basis of $\overline{\overline{C}}$). According to the previous proposition, $\overline{\overline{F}}$ admits a **unique** decomposition into a rotation and a pure deformation, see Figure 4.12.

$$\boxed{\overline{\overline{F}} = \overline{\overline{R}}.\overline{\overline{V}}} \tag{4.14}$$

The tensor $\overline{\overline{R}}$ represents a rotation that rotates the vectors of the basis $\left(\overrightarrow{U_i}\right)_{i=1,2,3}$; $\overline{\overline{R}}^T.\overline{\overline{R}} = \overline{\overline{I}}$.

The symmetric tensor $\overline{\overline{V}}$ merely stretches the eigenvectors of the basis $\left(\overrightarrow{U_i}\right)_{i=1,2,3}$. This is called a pure deformation.

4.4 Strain Tensors

4.4.1 Green-Lagrange Tensor

The intuitive notion of deformation of a system arises from the comparison between an initial configuration and a final configuration. Intuitively, one may say that a material system does not deform between t_0 and t if distances and angles are preserved. The mathematical tool that allows us to quantify these quantities is the scalar product. Consequently, comparing the scalar products of two vectors in κ_0 and then in κ_t allows us to define the notion of deformation.

Let $\overrightarrow{X}$ and $\overrightarrow{Y}$ be two vectors belonging to κ_0; these two vectors are transformed respectively into $\overrightarrow{x}$ and $\overrightarrow{y}$ in κ_t, see Figure 4.13.

$$\overrightarrow{x} \in \kappa_t \Rightarrow \overrightarrow{x} = \overline{\overline{F}}.\overrightarrow{X}$$
$$\overrightarrow{y} \in \kappa_t \Rightarrow \overrightarrow{y} = \overline{\overline{F}}.\overrightarrow{Y}$$

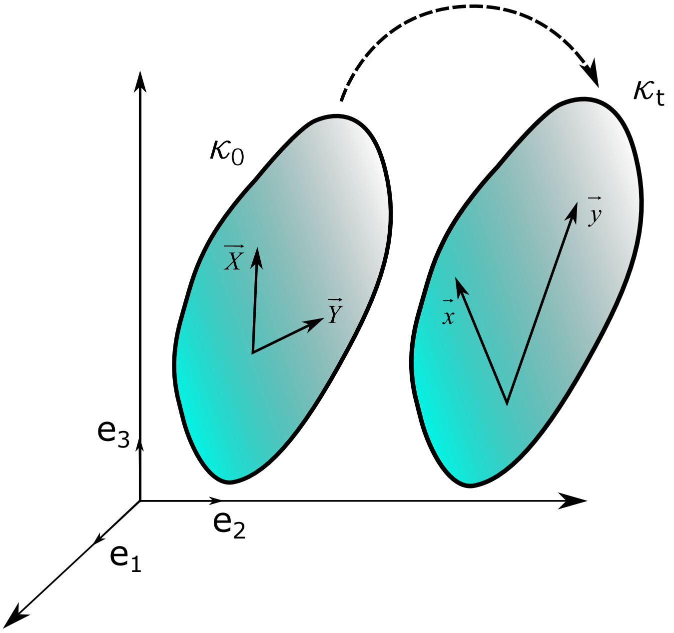


Figure 4.13: Transformation of the principal basis of the Cauchy stretch tensor.

The difference of the scalar products is

$$\begin{aligned}(\overrightarrow{x}\cdot\overrightarrow{y}) - \left(\overrightarrow{X}\cdot\overrightarrow{Y}\right) &= \left(\overline{\overline{F}}.\overrightarrow{X}\cdot\overline{\overline{F}}.\overrightarrow{Y}\right) - \left(\overrightarrow{X}\cdot\overrightarrow{Y}\right)\\ &= \left(\overrightarrow{X}.\overline{\overline{F}}^T.\overline{\overline{F}}.\overrightarrow{Y}\right) - \left(\overrightarrow{X}\cdot\overrightarrow{Y}\right)\\ &= \overrightarrow{X}.\left(\overline{\overline{C}}-\overline{\overline{I}}\right).\overrightarrow{Y}\end{aligned}$$

where $\overline{\overline{I}}$ denotes the identity tensor.

The tensor $2.\overline{\overline{L}} = \overline{\overline{C}} - \overline{\overline{I}}$ allows us to measure the variation of the scalar product of the two vectors during the transformation.

The tensor

$$\boxed{\overline{\overline{L}} = \frac{1}{2}\left(\overline{\overline{C}}-\overline{\overline{I}}\right)} \qquad (4.15)$$

is called the **Green-Lagrange strain tensor**.

> $\overline{\overline{L}}$ is a symmetric second-order tensor; it is therefore diagonalizable. Its eigenvalues are expressed in terms of those of $\overline{\overline{C}}$ by
>
> $$l_i = \frac{1}{2}\left(\lambda_i^2 - 1\right)$$

We now use the Green-Lagrange tensor $\overline{\overline{L}}$ to recalculate the difference in metric between the two configurations κ_0 and κ_t.

Unit Stretch

According to Equation 4.8, the unit stretch in the direction $\vec{\tau}$ is given by

$$\lambda\left(\tau\right) = \sqrt{\vec{\tau}.\overline{\overline{C}}.\vec{\tau}} - 1$$

Using 4.15, we replace $\overline{\overline{C}}$ by its value. We thus obtain

$$\boxed{\lambda\left(\vec{\tau}\right) = \sqrt{\overline{\overline{I}} + 2\vec{\tau}.\overline{\overline{L}}.\vec{\tau}} - 1} \tag{4.16}$$

Shear Between Two Directions

Equations 4.9 and 4.15 give

$$\boxed{\sin\left(\gamma\right) = \frac{2\vec{\tau}'.\overline{\overline{L}}.\vec{\tau}}{\sqrt{1+2\vec{\tau}.\overline{\overline{L}}.\vec{\tau}}\sqrt{1+2\vec{\tau}'.\overline{\overline{L}}.\vec{\tau}'}}} \tag{4.17}$$

Volumetric Stretch

The volumetric stretch is given by

$$dv = J dv_0$$

with $J = \sqrt{\det(\overline{\overline{C}})}$. Equation 4.15 yields

$$\boxed{dv = \sqrt{\det(\overline{\overline{I}} + 2\overline{\overline{L}})} dv_0} \tag{4.18}$$

> Why do we use the tensor $\overline{\overline{L}}$ instead of $\overline{\overline{C}}$ to describe the deformation of a continuum medium? The answer is related to rigid-body motion. According to the polar decomposition, and in the absence of stretch, one has
>
> $$\overline{\overline{F}} = \overline{\overline{R}}$$
>
> hence
>
> $$\overline{\overline{C}} = \overline{\overline{R}}^T.\overline{\overline{R}} = \overline{\overline{I}}$$
>
> Now, the strain in a rigid-body motion is zero, which $\overline{\overline{C}}$ does not directly reflect. This property is obviously satisfied by the tensor $\overline{\overline{L}}$.

4.5 Formulation in Terms of Displacements

4.5.1 Displacements

Definition 4.5.1 — Displacement. The displacement of the particle M, located at M_0 with coordinates $\overrightarrow{X}$ in κ_0 and at M_t with coordinates $\overrightarrow{x}$ in κ_t, is the vector $\overrightarrow{\xi}$ defined by

$$\boxed{\overrightarrow{\xi}\left(\overrightarrow{X},t\right) = \overrightarrow{M_0M_t} = \overrightarrow{x} - \overrightarrow{X}} \tag{4.19}$$

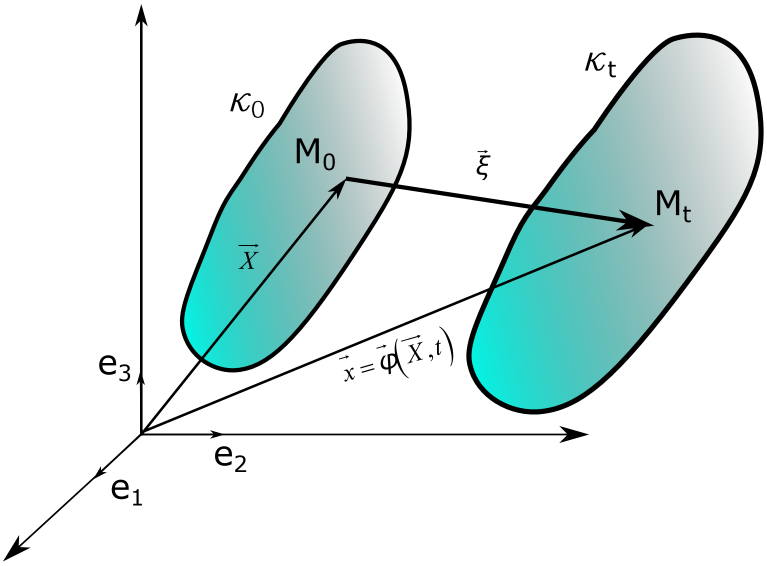


Figure 4.14: Displacement of a material point.

A displacement vector is defined for each particle M of the continuum medium; it is therefore a **vector field** defined on κ_0.
It is often more convenient to express the tensors introduced so far in terms of the displacement vector.

4.5.2 The Transformation Gradient Tensor

According to Equations 4.3 and 4.19, we have

$$\overrightarrow{\phi}\left(\overrightarrow{X},t\right) = \overrightarrow{x}$$

$$= \overrightarrow{\xi} + \overrightarrow{X}$$

Now,

$$\overline{\overline{F}} = \overline{\overline{\nabla\phi}}$$

Hence

$$\boxed{\overline{\overline{F}} = \overline{\overline{I}} + \overline{\overline{\nabla\xi}}} \tag{4.20}$$

4.5.3 The Cauchy-Green Stretch Tensor

The tensor $\overline{\overline{C}}$ is given by

$$\overline{\overline{C}} = \overline{\overline{F}}^T . \overline{\overline{F}}$$

Using Equation 4.20, one obtains

$$\boxed{\overline{\overline{C}} = \overline{\overline{I}} + \overline{\overline{\nabla\xi}} + \overline{\overline{\nabla\xi}}^T + \overline{\overline{\nabla\xi}}^T . \overline{\overline{\nabla\xi}}} \tag{4.21}$$

4.5.4 The Green-Lagrange Strain Tensor

The tensor $\overline{\overline{L}}$ is given by

$$\overline{\overline{L}} = \frac{1}{2}\left(\overline{\overline{C}} - \overline{\overline{I}}\right)$$

Using Equation 4.21, one obtains

$$\boxed{\overline{\overline{L}} = \frac{1}{2}(\overline{\overline{\nabla\xi}} + \overline{\overline{\nabla\xi}}^T + \overline{\overline{\nabla\xi}}^T . \overline{\overline{\nabla\xi}})} \tag{4.22}$$

4.6 Small-Strain Assumption

Most structures of engineering interest undergo small strains, even when they experience large displacements.

Definition 4.6.1 — Small-Strain Assumption. Strains between times 0 and t are said to be small at M_0 if, at that point, the eigenvalues of the tensor $\overline{\overline{L}}$ are small compared with 1 $(l_i \ll 1)$. Let l_i denote the eigenvalues of $\overline{\overline{L}}$. Equation 4.15 then gives

$$\boxed{l_i = \frac{1}{2}\left(\lambda_i^2 - 1\right) \Leftrightarrow \lambda_i = (2l_i + 1)^{\frac{1}{2}} \simeq l_i + 1} \tag{4.23}$$

4.6.1 Consequences

Unit stretch in the direction $\overrightarrow{\tau}$:

$$\lambda\left(\overrightarrow{\tau}\right) = \overrightarrow{\tau} . \overline{\overline{L}} . \overrightarrow{\tau}$$

Evolution of volumes:

$$J = \left(\det\left(\overline{\overline{I}} + 2\overline{\overline{L}}\right)\right)^{\frac{1}{2}} \simeq 1 + tr\left(\overline{\overline{L}}\right)$$

4.7 Small-Transformation Assumption

Definition 4.7.1 — Small-Transformation Assumption. Transformations between times 0 and t are said to be small at M_0 if, at that point, all components of the tensor $\overline{\overline{\nabla\xi}}\left(\overrightarrow{X}, t\right)$ are small compared with 1:

$$\boxed{\left\|\overline{\overline{\nabla\xi}}\left(\overrightarrow{X}, t\right)\right\| \ll 1 \quad \forall M_0 \in \kappa_0} \tag{4.24}$$

4.7.1 Consequences

- The eigenvalues of $\overline{\overline{L}}$ are also small. The small-transformation assumption implies the small-strain assumption. The converse is not always true.
- Since $\left\|\overline{\overline{\nabla\xi}}\left(\overrightarrow{X},t\right)\right\|$ is taken as the principal infinitesimal quantity, the quadratic terms $\overline{\overline{\nabla\xi}}^T.\overline{\overline{\nabla\xi}}$ in the expression of $\overline{\overline{L}}$ may be neglected. This leads to the linearization process:

$$\boxed{\overline{\overline{L}} = \overline{\overline{\varepsilon}} = \frac{1}{2}(\overline{\overline{\nabla\xi}} + \overline{\overline{\nabla\xi}}^T)} \tag{4.25}$$

$\overline{\overline{\varepsilon}}$ is the symmetric linearized strain tensor.
One also defines the antisymmetric tensor $\overline{\overline{\varpi}}$ by

$$\boxed{\overline{\overline{\varpi}} = \frac{1}{2}(\overline{\overline{\nabla\xi}} - \overline{\overline{\nabla\xi}}^T)} \tag{4.26}$$

- Relationship between $\overline{\overline{U}}$ and $\overline{\overline{\varepsilon}}$:
$$\overline{\overline{U}} \approx \overline{\overline{I}} + \overline{\overline{\varepsilon}}$$
- Unit stretch in the direction $\overrightarrow{\tau}$:
$$\lambda\left(\overrightarrow{\tau}\right) \approx \overrightarrow{\tau}.\overline{\overline{\varepsilon}}.\overrightarrow{\tau}$$
- Evolution of volumes:
$$J \simeq 1 + tr\left(\overline{\overline{\varepsilon}}\right) \simeq 1 + div\left(\overrightarrow{\xi}\right)$$

4.8 Compatibility Conditions

In this paragraph, we study the conditions that a strain field must satisfy in order to be integrable and to give rise to a continuous displacement field. Imagine a puzzle. If the pieces of the puzzle are deformed independently, nothing guarantees that they can be assembled after deformation without gaps or overlaps. The strain field must therefore satisfy certain conditions in order to correspond to a continuous displacement field. These conditions are called compatibility conditions.

4.8.1 Uniqueness of the Displacement Field

Before studying compatibility conditions, let us examine the uniqueness of the displacement field. Consider the strains under the assumption of small transformations.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\xi_i}{\partial X_j} + \frac{\partial\xi_j}{\partial X_i}\right)$$

The components of the displacement gradient tensor and the components of the small-rotation tensor are written respectively as

$$H_{ij} = \nabla\xi_{ij} = \frac{\partial\xi_i}{\partial X_j} \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial\xi_i}{\partial X_j} - \frac{\partial\xi_j}{\partial X_i}\right)$$

The derivatives of the components of the rotation tensor can be expressed in terms of the differences of the small strains. Indeed, consider the partial derivatives of the components of the rotation tensor:

$$\frac{\partial\omega_{ij}}{\partial X_k} = \frac{1}{2}\left(\frac{\partial^2\xi_i}{\partial X_j\partial X_k} - \frac{\partial^2\xi_j}{\partial X_i\partial X_k}\right)$$

By adding and subtracting the term involving $\partial^2\xi_k/(\partial X_i \partial X_j)$, the previous expression can be rewritten in the form

$$\frac{\partial \omega_{ij}}{\partial X_k} = \frac{1}{2}\left(\frac{\partial^2 \xi_i}{\partial X_j \partial X_k} + \frac{\partial^2 \xi_k}{\partial X_i \partial X_j}\right) - \frac{1}{2}\left(\frac{\partial^2 \xi_k}{\partial X_i \partial X_j} + \frac{\partial^2 \xi_j}{\partial X_i \partial X_k}\right)$$

Assuming that the displacement field is twice continuously differentiable, we may interchange the order of partial differentiation and make the derivatives of the strain tensor components appear:

$$\frac{\partial \omega_{ij}}{\partial X_k} = \frac{\partial}{\partial X_j}\left\{\frac{1}{2}\left(\frac{\partial \xi_i}{\partial X_k} + \frac{\partial \xi_k}{\partial X_i}\right)\right\} - \frac{\partial}{\partial X_i}\left\{\frac{1}{2}\left(\frac{\partial \xi_k}{\partial X_j} + \frac{\partial \xi_j}{\partial X_k}\right)\right\}$$

thus

$$\frac{\partial \omega_{ij}}{\partial X_k} = \frac{\partial \varepsilon_{ik}}{\partial X_j} - \frac{\partial \varepsilon_{jk}}{\partial X_i}$$

If the strains are zero $\left(\overline{\overline{\varepsilon}} = 0\right)$, then the rotation tensor $\overline{\overline{\varpi}}$ is constant and the displacement gradient tensor $\overline{\overline{H}}$ is equal to the rotation tensor $\overline{\overline{\varpi}}$:

$$\overline{\overline{\varepsilon}} = 0 \Rightarrow \overline{\overline{\varpi}} = \overline{\overline{Cst}}, \quad \overline{\overline{H}} = \overline{\overline{\varpi}}$$

The displacement field corresponding to a translation $\overrightarrow{\xi_0}$ and a rigid-body rotation is

$$H_{ij} = \frac{\partial \xi_i}{\partial X_j} = \omega_{ij}(t) \Rightarrow \xi_i = \omega_{ij}(t) X_j + \xi_0$$

Thus, the knowledge of the strain field determines the displacement field up to a rigid-body rotation and a translation.

4.8.2 Existence of a Displacement Field: Compatibility Conditions

$$\boxed{\frac{\partial \omega_{ij}}{\partial X_k} = \frac{\partial \varepsilon_{ik}}{\partial X_j} - \frac{\partial \varepsilon_{jk}}{\partial X_i}} \tag{4.27}$$

A sufficient condition for this differential system to be integrable is

$$\frac{\partial^2 \omega_{ij}}{\partial X_k \partial X_l} = \frac{\partial^2 \omega_{ij}}{\partial X_l \partial X_k} \, (l \neq k)$$

The corresponding equation for the components of the strain tensor is

$$\boxed{\frac{\partial^2 \varepsilon_{ik}}{\partial X_j \partial X_l} - \frac{\partial^2 \varepsilon_{jk}}{\partial X_i \partial X_l} = \frac{\partial^2 \varepsilon_{il}}{\partial X_j \partial X_k} - \frac{\partial^2 \varepsilon_{jl}}{\partial X_i \partial X_k}} \tag{4.28}$$

Taking symmetries into account, there remain 6 independent relations to be satisfied. The preceding equations ensure that the system of Equations 4.27 is integrable. One can therefore determine the rotation tensor $\overline{\overline{\varpi}}$ if the strain tensor $\overline{\overline{\varepsilon}}$ is known. Thus, the displacement gradient tensor $\overline{\overline{H}}$ is defined. The question now is whether one can recover the displacements from the knowledge of this tensor. In other words, is the system

$$\frac{\partial \xi_i}{\partial X_j} = \varepsilon_{ij} + \omega_{ij}$$

integrable? The answer is positive if

$$\frac{\partial^2 \xi_i}{\partial X_j \partial X_l} = \frac{\partial^2 \xi_i}{\partial X_l \partial X_j} (l \neq j)$$

that is,

$$\frac{\partial}{\partial X_l}(\varepsilon_{ij} + \omega_{ij}) = \frac{\partial}{\partial X_j}(\varepsilon_{il} + \omega_{il})(l \neq j)$$

This equality is indeed satisfied according to Relations 4.27. The passage from $\overline{\overline{H}}$ to the displacement field therefore introduces no additional compatibility conditions. The 6 independent relations among Equations 4.28 can be written as

$$\boxed{\begin{aligned}
\frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} &= \frac{\partial}{\partial X_1}\left\{\frac{\partial \varepsilon_{13}}{\partial X_2} + \frac{\partial \varepsilon_{21}}{\partial X_3} - \frac{\partial \varepsilon_{32}}{\partial X_1}\right\} \\
\frac{\partial^2 \varepsilon_{22}}{\partial X_1 \partial X_3} &= \frac{\partial}{\partial X_2}\left\{\frac{\partial \varepsilon_{21}}{\partial X_3} + \frac{\partial \varepsilon_{32}}{\partial X_1} - \frac{\partial \varepsilon_{13}}{\partial X_2}\right\} \\
\frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} &= \frac{\partial}{\partial X_3}\left\{\frac{\partial \varepsilon_{32}}{\partial X_1} + \frac{\partial \varepsilon_{13}}{\partial X_2} - \frac{\partial \varepsilon_{21}}{\partial X_3}\right\} \\
2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} &= \frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} \\
2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3} &= \frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} \\
2\frac{\partial^2 \varepsilon_{31}}{\partial X_3 \partial X_1} &= \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial X_3^2}
\end{aligned}} \tag{4.29}$$

The compatibility relations can be written in a more compact form if Gibbs notation is adopted. In Einstein index notation, the compatibility relations are

$$\underbrace{\frac{\partial^2}{\partial X_k \partial X_k}\varepsilon_{ij}}_{\Delta\varepsilon_{ij}} + \underbrace{\frac{\partial^2}{\partial X_i \partial X_j}\overbrace{\varepsilon_{kk}}^{trace(\overline{\overline{\varepsilon}})}}_{\overrightarrow{grad}\left\{\overrightarrow{grad}\left[trace(\overline{\overline{\varepsilon}})\right]\right\}} - \underbrace{\frac{\partial}{\partial X_j}\left(\frac{\partial}{\partial X_k}\varepsilon_{ik}\right)}_{\overrightarrow{grad}\left[div(\overline{\overline{\varepsilon}})\right]} - \frac{\partial^2}{\partial X_i \partial X_k}\varepsilon_{jk} = 0$$

which leads to the following expression in Gibbs notation:

$$\boxed{\Delta\left(\overline{\overline{\varepsilon}}\right) + \overrightarrow{grad}\left\{\overrightarrow{grad}\left[trace\left(\overline{\overline{\varepsilon}}\right)\right]\right\} = \overrightarrow{grad}\left[div\left(\overline{\overline{\varepsilon}}\right)\right] + \left\{\overrightarrow{grad}\left[div\left(\overline{\overline{\varepsilon}}\right)\right]\right\}^T} \tag{4.30}$$

4.9 Exercise

4.9.1 Statement

Consider a motion defined by the equations

$$\begin{cases} x_1 = X_1 + \alpha.t.X_2 \\ x_2 = X_2 + \alpha.t.X_1 \\ x_3 = X_3 \end{cases}$$

with $\alpha > 0$.

Assume that the reference configuration Ω_0 is a volume of particles occupying a cube of side length $2l$ centered at O.

1. Compute, for every time t, the volume $\gamma(t)$ of the domain Ω constituting the deformed configuration.

2. Plot the curve $\gamma(t)$ as a function of time for $\alpha = 1$. Deduce the time t_* beyond which the motion ceases to be physical.
3. Compute the relative stretch around an arbitrary point X of ω_0 in direction 1 generated by the vector e_1.
4. Compute the shear angle $\gamma_{12}(t)$.
5. Compute the relative stretch around an arbitrary point X of ω_0 in the directions generated by the vectors $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$.
6. Compute the shear angle between these two directions.

4.9.2 Solution

- Compute, for every time t, the volume $\gamma(t)$ of the domain Ω constituting the deformed configuration.

Computation of the transformation matrix $\overline{\overline{F}}$:

$$\overline{\overline{F}} = \begin{bmatrix} 1 & \alpha.t & 0 \\ \alpha.t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The volume $\gamma(t)$ of the domain Ω constituting the deformed configuration is

$$\gamma(t) = J.\gamma(0)$$

with

$$J = det(\overline{\overline{F}})$$

$$J = \left(1 - \alpha^2.t^2\right)$$

$$\gamma(0) = 8.l^3$$

- Plot the curve $\gamma(t)$ as a function of time for $\alpha = 1$. Deduce the time t_* beyond which the motion ceases to be physical.

$$\alpha = 1$$

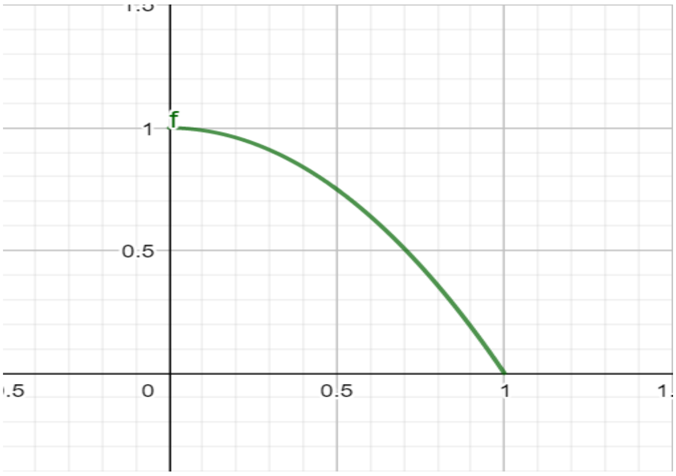


Figure 4.15: Variation of the volume $\gamma(t)$ as a function of time.

The time t_* beyond which the motion ceases to be physical is obtained from

$$J > 0$$

$$J = \left(1 - \alpha^2.t^2\right) > 0$$

$$\Rightarrow t < \frac{1}{\alpha}$$

- Compute the relative stretch around an arbitrary point X of ω_0 in direction 1 generated by the vector e_1.

Computation of the stretch tensor:

$$\overline{\overline{C}} = \overline{\overline{F}}^T.\overline{\overline{F}}$$

$$\overline{\overline{C}} = \begin{bmatrix} 1 & \alpha.t & 0 \\ \alpha.t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & \alpha.t & 0 \\ \alpha.t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overline{\overline{C}} = \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The relative stretch around an arbitrary point X of ω_0 in direction 1 generated by the vector e_1 is

$$\frac{l}{l_0} = \sqrt{\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_1}}$$

with

$$\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\frac{l}{l_0} = \sqrt{1+\alpha^2.t^2}$$

- Compute the shear angle $\gamma_{12}(t)$.

$$\sin(\gamma_{12}) = \frac{\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_2}}{\sqrt{\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_1}}.\sqrt{\overrightarrow{e_2}.\overline{\overline{C}}.\overrightarrow{e_2}}}$$

$$\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_2} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{e_1}.\overline{\overline{C}}.\overrightarrow{e_2} = 2.\alpha.t$$

$$\overrightarrow{e_2}.\overline{\overline{C}}.\overrightarrow{e_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{e_2}.\overline{\overline{C}}.\overrightarrow{e_2} = 1+\alpha^2.t^2$$

$$\gamma_{12} = \text{Arcsin}\left(\frac{2.\alpha.t}{1+\alpha^2.t^2}\right)$$

- Compute the relative stretch around an arbitrary point X of ω_0 in the directions generated by the vectors $\overrightarrow{v_1} = \overrightarrow{e_1} + \overrightarrow{e_2}$ and $\overrightarrow{v_2} = \overrightarrow{e_1} - \overrightarrow{e_2}$.

Normalize the vector $\overrightarrow{v_1}$:

$$\overrightarrow{v^*{}_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

The relative stretch around an arbitrary point X of ω_0 in the direction generated by $\overrightarrow{v_1}$ is

$$\frac{l}{l_0} = \sqrt{\overrightarrow{v^*{}_1}.\overline{\overline{C}}.\overrightarrow{v^*{}_1}}$$

$$\overrightarrow{v^*{}_1}.\overline{\overline{C}}.\overrightarrow{v^*{}_1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\frac{l}{l_0} = 1+\alpha.t$$

Normalize the vector $\overrightarrow{v_2}$:

$$\overrightarrow{v^*{}_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

The relative stretch around an arbitrary point X of ω_0 in the direction generated by $\overrightarrow{v_2}$ is

$$\frac{l}{l_0} = \sqrt{\overrightarrow{v^*{}_2}.\overline{\overline{C}}.\overrightarrow{v^*{}_2}}$$

$$\overrightarrow{v^*{}_2}.\overline{\overline{C}}.\overrightarrow{v^*{}_2} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

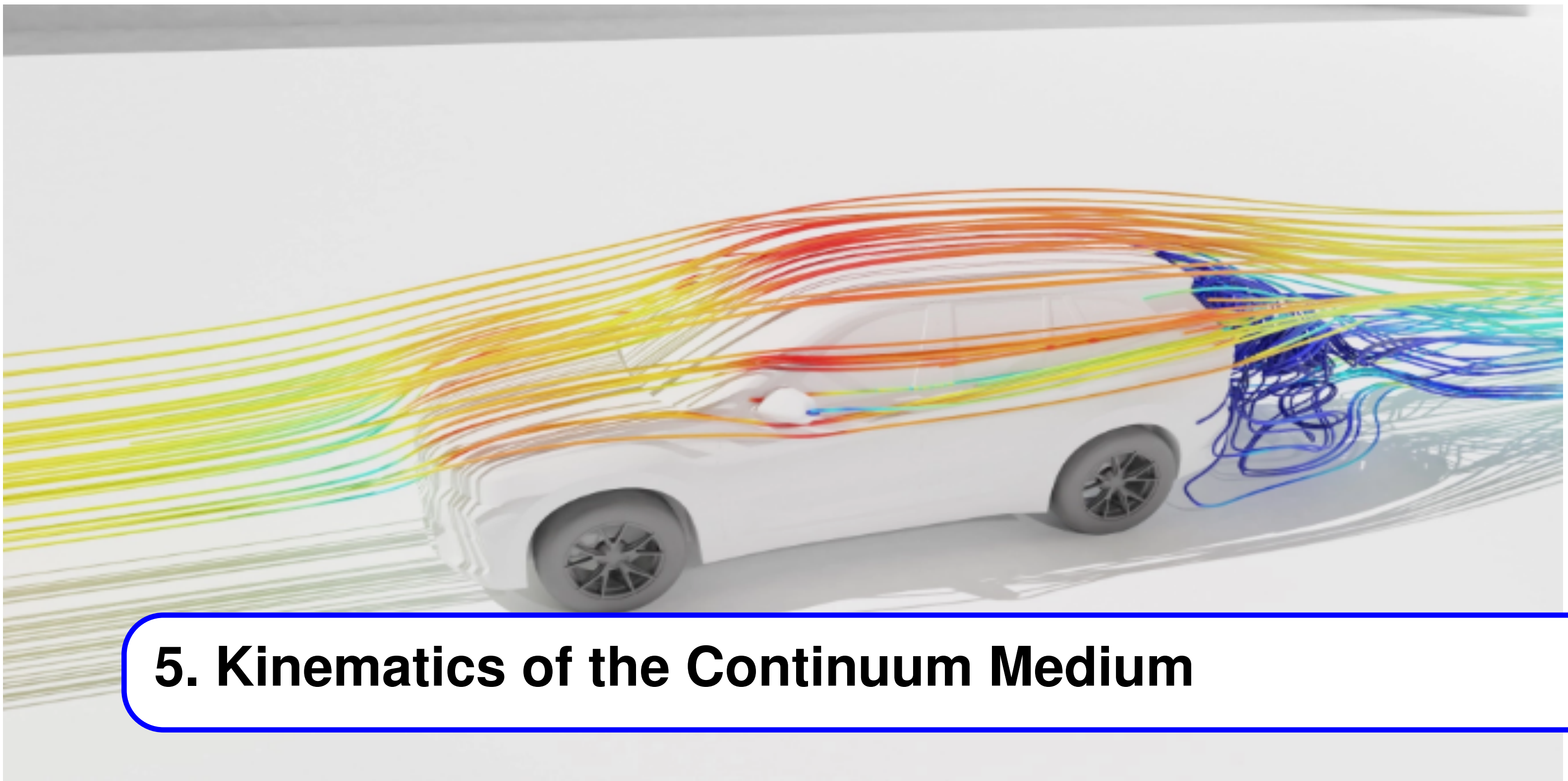
$$\frac{l}{l_0} = 1-\alpha.t$$

- Compute the shear angle between these two directions.

$$\sin(\gamma) = \frac{\overrightarrow{v^*{}_1}.\overline{\overline{C}}.\overrightarrow{v^*{}_2}}{\sqrt{\overrightarrow{v^*{}_1}.\overline{\overline{C}}.\overrightarrow{v^*{}_1}}.\sqrt{\overrightarrow{v^*{}_2}.\overline{\overline{C}}.\overrightarrow{v^*{}_2}}}$$

$$\overrightarrow{v^*}_1.\overline{\overline{C}}.\overrightarrow{v^*}_2 = \begin{bmatrix} {}^{1}\!\big/\!{\sqrt{2}} & {}^{1}\!\big/\!{\sqrt{2}} & 0 \end{bmatrix} . \begin{bmatrix} 1+\alpha^2.t^2 & 2.\alpha.t & 0 \\ 2.\alpha.t & 1+\alpha^2.t^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} {}^{1}\!\big/\!{\sqrt{2}} \\ -{}^{1}\!\big/\!{\sqrt{2}} \\ 0 \end{bmatrix} = 0$$

$$\sin(\gamma) = 0$$



5. Kinematics of the Continuum Medium

5.1 Introduction

In the previous chapter, we focused on the comparison between the reference configuration and the current configuration, essentially from a geometric point of view, without considering the intermediate history of the system under study. As stated there, the argument t appeared in the various expressions as the parameter indexing the current configuration.

We now aim to follow the evolution of the system as a function of time. We shall first be concerned with the purely geometric aspect of this evolution: this is the study of the kinematics of the three-dimensional continuum medium. From this, we shall examine other aspects of the evolution, in particular by considering the physical quantities attached to the system or to its constituent elements, in their Eulerian representations, as well as the time derivatives of these quantities.

In a suggestive form, referring to the initial discussion of the previous chapter, one could describe the purpose of this chapter as the tracking of the "film" of the system evolution by comparing its images "frame by frame."

Definition 5.1.1 — Eulerian Description. The Eulerian description provides an incremental description of the evolution of a continuum medium. It mainly consists in following the evolution of the system step by step in time. In other words, once the current configuration κ_t has been determined, it is taken as the reference configuration for computing the configuration κ_{t+dt}.

5.2 Velocity Gradient Tensor

In order to describe the kinematics of the continuum medium, the equations are written in terms of velocity. Let $\overrightarrow{dx}$ be an elementary vector belonging to the current configuration κ_t. We compute the time variation of $\overrightarrow{dx}$, namely $\dot{\overrightarrow{dx}}$, during the transition from κ_t to κ_{t+dt}.

To do so, let us use, **artificially**, the initial configuration κ_0. According to Equation 4.7:

$$\overrightarrow{dx} = \overline{\overline{F}}.\overrightarrow{dX}$$

hence:

$$\boxed{\dot{\overrightarrow{dx}} = \frac{d\left(\overline{\overline{F}}.\overrightarrow{dX}\right)}{dt} = \frac{d\left(\overline{\overline{\nabla\phi}}.\overline{\overline{F}}^{-1}.\overrightarrow{dx}\right)}{dt} = \overline{\overline{\nabla}}\left(\frac{d\phi}{dt}\right).\overline{\overline{F}}^{-1}.\overrightarrow{dx} = \overline{\overline{\nabla V}}(\overrightarrow{X},t)\overline{\overline{F}}^{-1}.\overrightarrow{dx}} \tag{5.1}$$

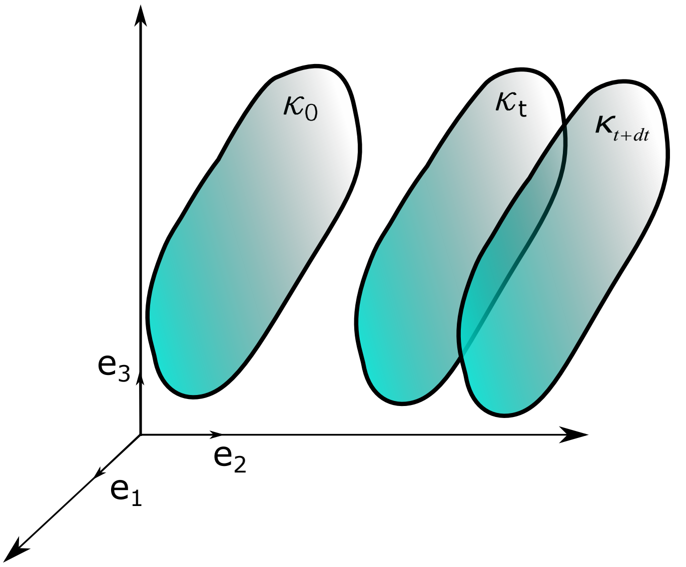


Figure 5.1: Kinematics of the continuum medium.

Now one has

$$\vec{V}\left(\vec{X},t\right) = \vec{V}\left(\vec{x},t\right)$$

hence:

$$\overline{\overline{\nabla V}}\left(\vec{X},t\right).\overrightarrow{dX} = \overline{\overline{grad}}\vec{V}\left(\vec{x},t\right).\overrightarrow{dx}$$

$$= \overline{\overline{grad}}\vec{V}\left(\vec{x},t\right).\overline{\overline{F}}.\overrightarrow{dX}$$

Multiplying the previous equation on both sides by $\overline{\overline{F}}^{-1}$, one obtains:

$$\boxed{\overline{\overline{\nabla V}}\left(\vec{X},t\right).\overline{\overline{F}}^{-1} = \overline{\overline{grad}}\vec{V}\left(\vec{x},t\right)} \tag{5.2}$$

Finally, substituting Equation 5.2 into Equation 5.1, one obtains:

$$\boxed{\dot{\overrightarrow{dx}} = \overline{\overline{grad}}\vec{V}\left(\vec{x},t\right).\overrightarrow{dx}} \tag{5.3}$$

Definition 5.2.1 — Velocity Gradient Tensor. $\overline{\overline{grad}}\vec{V}\left(\vec{x},t\right)$ is the Eulerian velocity gradient tensor. This tensor describes the instantaneous transition from the current configuration κ_t to the configuration κ_{t+dt}. In the Eulerian description, it plays the same role as the deformation gradient tensor $\overline{\overline{F}}$ in the Lagrangian description.

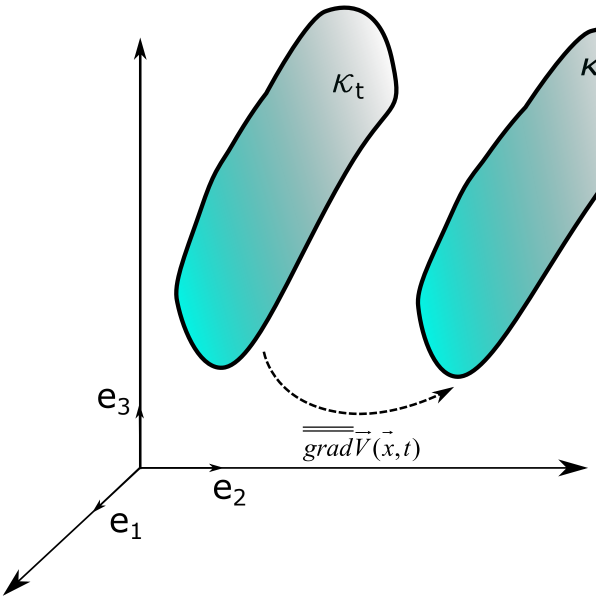


Figure 5.2: Definition of the velocity gradient tensor.

5.3 Strain-Rate Tensor

As in the Lagrangian description, and starting from Formula 5.3, one may compute the material derivative of the scalar product of two elementary vectors $\overrightarrow{dx}$ and $\overrightarrow{dx'}$ belonging to κ_t.

$$
\begin{aligned}
&\dot{\overrightarrow{dx'}}.\overrightarrow{dx} = \dot{\overrightarrow{dx}}.\overrightarrow{dx'} + \overrightarrow{dx}.\dot{\overrightarrow{dx'}}\\
&= \overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right).\overrightarrow{dx}.\overrightarrow{dx'} + \overrightarrow{dx}.\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right).\overrightarrow{dx'}\\
&= \overrightarrow{dx}.{}^t\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right).\overrightarrow{dx'} + \overrightarrow{dx}.\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right).\overrightarrow{dx'}\\
&= \overrightarrow{dx}.\left({}^t\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right) + \overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right)\right).\overrightarrow{dx'}\\
&= 2.\overrightarrow{dx}.\overline{\overline{D}}.\overrightarrow{dx'}
\end{aligned}
\tag{5.4}
$$

with

$$
\boxed{D = \frac{1}{2}.\left(\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right) + {}^T\overline{\overline{grad}}\overrightarrow{V}\left(\overrightarrow{x},t\right)\right)}
\tag{5.5}
$$

Definition 5.3.1 — Strain-Rate Tensor. $\overline{\overline{D}}$ is the strain-rate tensor. It is the symmetric part of the velocity gradient tensor. The tensor $\overline{\overline{D}}$ characterizes the evolution of the metric, that is, the deformation referred to the current configuration, which therefore plays, at each instant, the role of reference configuration.

5.3.1 Rate of Change of Lengths

As in the Lagrangian description, we now compute the rate of change of the metric of the continuum medium. Let $\overrightarrow{dx}$ be an elementary vector of unit direction $\overrightarrow{\tau}$ and length l.

$$\overrightarrow{dx} = l\,\overrightarrow{\tau}$$

By squaring the norm of this vector and differentiating with respect to time, one obtains:

$$\overrightarrow{dx}.\overrightarrow{dx} = l^2 \Rightarrow \dot{\overrightarrow{dx}}.\overrightarrow{dx} = 2.l.\dot{l}$$

Now, from (5.4):

$$\dot{\overrightarrow{dx}}.\overrightarrow{dx} = 2.\overrightarrow{dx}.\overline{\overline{D}}.\overrightarrow{dx} = 2.l.\overrightarrow{\tau}.\overline{\overline{D}}.l.\overrightarrow{\tau}$$

Equating the two previous expressions gives the rate of change of l:

$$\boxed{\frac{\dot{l}}{l} = \overrightarrow{\tau}.\overline{\overline{D}}.\overrightarrow{\tau}} \tag{5.6}$$

5.3.2 Rate of Shear of Two Directions

Consider two directions defined by the unit vectors $\overrightarrow{\tau}$ and $\overrightarrow{\tau}'$, orthogonal at the point $\overrightarrow{x}$ at time t. At time $t' = t + dt$, these two vectors form an angle equal to $\left(\frac{\pi}{2} - \gamma\right)$, see Figure 5.3.

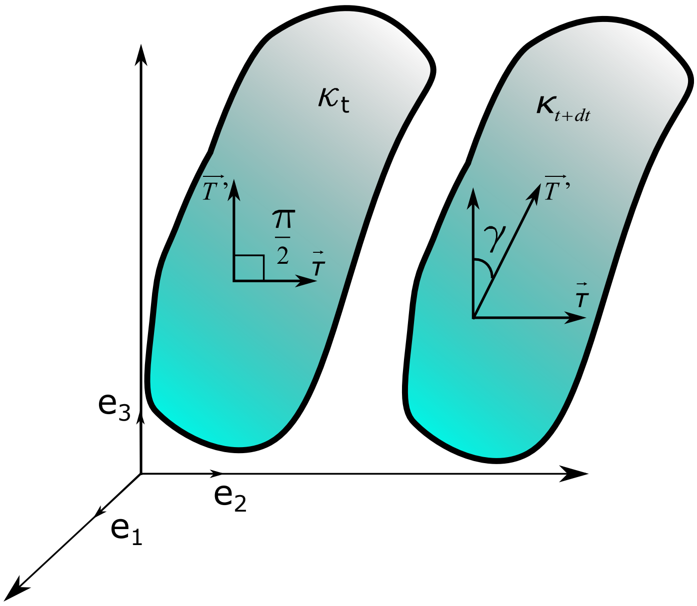


Figure 5.3: Rate of shear.

Now,

$$\overrightarrow{\tau}\cdot\overrightarrow{\tau}' = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin(\gamma)$$

Since $\gamma = 0$ at time t, one obtains

$$\boxed{\dot{\gamma} = 2\,\overrightarrow{\tau}.\overline{\overline{D}}.\overrightarrow{\tau}'} \tag{5.7}$$

5.3.3 Rate of Change of Volume

The rate of change of volume is given by

$$\boxed{\frac{\dot{dv}}{dv} = div\overrightarrow{V} = \frac{\dot{J}}{J}} \tag{5.8}$$

R In the case of an incompressible fluid, the rate of change of volume is zero; hence $div\overrightarrow{V} = 0$.

5.4 Rotation-Rate Tensor

Since the tensor $\overline{\overline{D}}$ has been defined as the symmetric part of the velocity gradient tensor $\overline{\overline{grad}}\overrightarrow{V}$, its antisymmetric part is given by

$$\boxed{\overline{\overline{\Omega}} = \frac{1}{2}\left(\overline{\overline{grad}}\overrightarrow{V}(\overrightarrow{x},t) - {}^{T}\overline{\overline{grad}}\overrightarrow{V}(\overrightarrow{x},t)\right)} \tag{5.9}$$

To this tensor, one may associate a dual vector $\overrightarrow{\Omega}$ such that

$$\overline{\overline{\Omega}}.\overrightarrow{u} = \overrightarrow{\Omega} \wedge \overrightarrow{u}$$

One shows that

$$\boxed{\overrightarrow{\Omega} = \frac{1}{2}\overrightarrow{rotV}} \tag{5.10}$$

The vector $\overrightarrow{\Omega}$ is called the rotation-rate vector or the vorticity vector.[1]

R From Equation 5.3, one has

$$\dot{\overrightarrow{dx}} = \overline{\overline{grad}}\overrightarrow{V}(\overrightarrow{x},t).\overrightarrow{dx} = (\overline{\overline{D}}+\overline{\overline{\Omega}}).\overrightarrow{dx} = \overline{\overline{D}}.\overrightarrow{dx} + \overrightarrow{\Omega} \wedge \overrightarrow{dx}$$

Let $\overrightarrow{dx_i}, i = 1,2,3$ be the three principal directions of $\overline{\overline{D}}$, and $d_i, i = 1,2,3$ the associated eigenvalues. In this case one has

$$\dot{\overrightarrow{dx_i}} = d_i.\overrightarrow{dx_i} + \overrightarrow{\Omega} \wedge \overrightarrow{dx_i}, i = 1,2,3$$

The velocity $\dot{\overrightarrow{dx_i}}$ between t and $t+dt$ is therefore composed of a unit stretch $d_i dt$ and an instantaneous rotation $\overrightarrow{\Omega}dt$. This justifies the name given to $\overrightarrow{\Omega}$: rotation vector or vorticity.

---

[1] This notion is very useful, as you will see in fluid mechanics.

5.5 Material Derivative

Throughout the previous paragraphs, we have seen that it is sometimes difficult to refer to an initial configuration in order to study the motion of a continuum medium. Although this situation is obvious in fluid mechanics, the analysis requires the computation of the rates of change of quantities attached to particles in motion. For this purpose, a mathematical tool is defined: the material derivative, also called the particle derivative.

5.5.1 Intuitive Notion

The objective is to characterize the variation with time of a quantity attached to a particle or to a set of particles followed in their motion. At a given point ($\overrightarrow{x}$ fixed), the quantity varies with time, and at a given instant (t fixed), it depends on the point at which it is considered.

■ **Example 5.1** Imagine that you are traveling by car from Sétif to Algiers and that you wish to know the outside temperature. The variation of temperature depends on two factors:

1. The temperature varies at the same location during the day; this is the unsteadiness marked by the dependence of T on t, with $\overrightarrow{x}$ fixed. For example, in Sétif, the morning temperature differs from the evening temperature: $T(09h) \neq T(16h) \Rightarrow T = T(t)$.
2. The fact that you are moving and passing through different places which, at the same instant, do not have the same temperature; this is the dependence of T on $\overrightarrow{x}$, with t fixed. It is clear that this spatial dependence ($\overrightarrow{x}$) depends on the speed at which you are traveling.

■

One may therefore schematically state that:

The variation of T = variation due to t + variation due to the motion, hence due to $\overrightarrow{x}$ (convective term).

5.5.2 Material Derivative of a Point Function

The quantity to be differentiated is here attached to a single moving particle $\overrightarrow{X}$. This particle occupies different geometric positions $\overrightarrow{x}$ over time.

Scalar Function:

$f = f(\overrightarrow{x},t) / \overrightarrow{x} = \overrightarrow{\phi}(\overrightarrow{X},t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}.\frac{\partial x_i}{\partial t}$$

hence

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \overrightarrow{grad} f.\overrightarrow{V}} \tag{5.11}$$

The material derivative is composed of two terms:

- $\frac{\partial f}{\partial t}$: variation of f at the geometric point $\overrightarrow{x}$ with respect to time. This term corresponds to the sole cause of variation of f when the particle is motionless at time t, that is, when its motion is such that $\overrightarrow{V} = 0$.
- $\overrightarrow{grad} f.\overrightarrow{V}$: term corresponding to the variation of f due to the motion of the particle, hence due to its convective transport by the motion $\left(\overrightarrow{V}\right)$. This term is the only cause of variation of f when the latter is independent of time, $\frac{\partial f}{\partial t} = 0$. This term is called the convection term.

R In the Lagrangian description, the material derivative is identified with the partial derivative with respect to time. Indeed, one has $f = f\left(\overrightarrow{X}, t\right) \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t}$ because $\overrightarrow{X}$ is always fixed. The formula established above is valid whatever the tensorial order of f.

Vector Function:
$f = \overrightarrow{f}\,(\overrightarrow{x}, t) / \overrightarrow{x} = \overrightarrow{\phi}\left(\overrightarrow{X}, t\right)$

$$\boxed{\frac{d\overrightarrow{f}}{dt} = \frac{\partial \overrightarrow{f}}{\partial t} + \overline{\overline{grad}}\,f.\overrightarrow{V}} \tag{5.12}$$

For each component, one has

$$\boxed{\frac{df_i}{dt} = f_{i,t} + f_{i,j}.V_j} \tag{5.13}$$

In the Eulerian description, the acceleration of a particle is given by the material derivative of the velocity:

$$\boxed{\overrightarrow{\gamma} = \frac{d\overrightarrow{V}}{dt} = \frac{\partial \overrightarrow{V}}{\partial t} + \overline{\overline{grad}}V.\overrightarrow{V}} \tag{5.14}$$

Second-Order Tensor Function:
$f = \overline{\overline{f}}(\overrightarrow{x}, t) / \overrightarrow{x} = \overrightarrow{\phi}\,(\overrightarrow{X}, t)$

$$\boxed{\frac{d\overline{\overline{f}}}{dt} = \frac{\partial \overline{\overline{f}}}{\partial t} + \overline{\overline{\overline{grad}\,\overline{\overline{f}}}}.\overrightarrow{V}} \tag{5.15}$$

For each component, one has

$$\boxed{\frac{df_{ij}}{dt} = f_{ij,t} + f_{ij,p}.V_P} \tag{5.16}$$

5.5.3 Material Derivative of a Volume Integral

We now consider the variation with respect to time of a quantity attached to a set of particles: a volume. Here $f\,(\overrightarrow{x}, t)$ is a volumetric density. Let the volume integral I be defined over a volume γ_t bounded by the frontier $\partial\gamma_t$, see Figure 5.4:

$$\boxed{I = \int\limits_{\gamma_t} f\,(\overrightarrow{x}, t)\, dv} \tag{5.17}$$

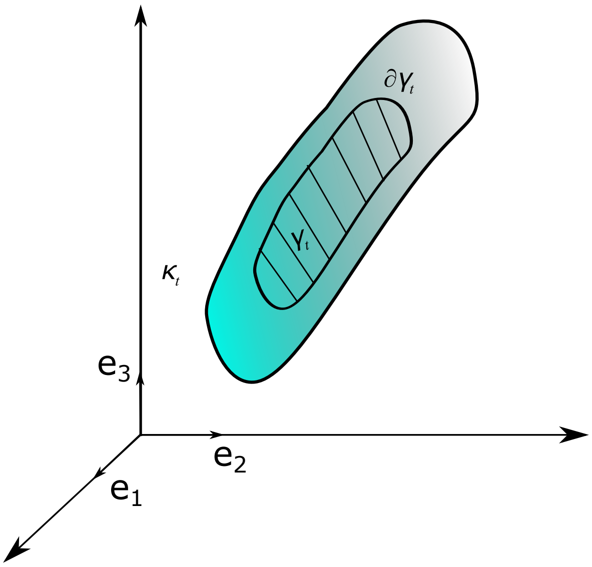


Figure 5.4: Material derivative of a volume integral.

■ **Example 5.2** If $f(\overrightarrow{x},t)$ represents the volumetric mass density (or density) $\rho(\overrightarrow{x},t)$, then the mass M of a volume γ_t is given by

$$M = \int\limits_{\gamma_t} \rho(\overrightarrow{x},t)\, dv$$

■

Our goal is to compute the variation with respect to time of the quantity I, namely

$$\boxed{\frac{dI}{dt} = \frac{d}{dt}\int\limits_{\gamma_t} f(\overrightarrow{x},t)\, dv} \tag{5.18}$$

The difficulty lies in the fact that the integration domain γ_t varies with time. To overcome this difficulty, one brings the integral back to the reference configuration, where the integration domain becomes fixed (γ_0). One then computes the derivative and afterwards returns to the Eulerian variables by performing the appropriate change of variables.

Case of a Continuous Function

Assume that $f(\overrightarrow{x},t)$ is continuous.

The variation of I is given by

$$\frac{dI}{dt} = \frac{d}{dt}\int\limits_{\gamma_t} f(\overrightarrow{x},t)\, dv$$

According to Equation (4.12), one has $dv = J dv_0$, hence, passing to the reference configuration,

$$\frac{dI}{dt} = \frac{d}{dt}\int\limits_{\gamma_0} f\left(\overrightarrow{X},t\right) J dv_0$$

Since the integration domain γ_0 is fixed, one may write

$$\frac{dI}{dt}=\int_{\gamma_0}\frac{d\left(f\left(\overrightarrow{X},t\right)J\right)}{dt}dv_0$$

$$\frac{dI}{dt}=\int_{\gamma_0}\left[\frac{df}{dt}J+\frac{dJ}{dt}f\right]dv_0$$

$$\frac{dI}{dt}=\int_{\gamma_0}\frac{df}{dt}Jdv_0+\int_{\gamma_0}\dot{J}fdv_0$$

Now, according to Formula 5.8, one has $\dot{J}=Jdiv\overrightarrow{V}$, hence

$$\frac{dI}{dt}=\int_{\gamma_0}\frac{df}{dt}Jdv_0+\int_{\gamma_0}fdiv\overrightarrow{V}Jdv_0$$

It now remains to express this relation in the current configuration in terms of Eulerian variables:

$$\frac{dI}{dt}=\int_{\gamma_t}\frac{df}{dt}dv+\int_{\gamma_t}fdiv\overrightarrow{V}dv$$

Using Formula 5.11 for the material derivative of f, one obtains

$$\frac{dI}{dt}=\int_{\gamma_t}\left[\frac{\partial f}{\partial t}+\overrightarrow{grad}f.\overrightarrow{V}+fdiv\overrightarrow{V}\right]dv$$

$$\frac{dI}{dt}=\int_{\gamma_t}\left[\frac{\partial f}{\partial t}+div\left(f\overrightarrow{V}\right)\right]dv$$

$$\frac{dI}{dt}=\int_{\gamma_t}\frac{\partial f}{\partial t}dv+\int_{\gamma_t}div\left(f\overrightarrow{V}\right)dv$$

Using the divergence theorem,[2] one obtains

$$\boxed{\frac{dI}{dt}=\int_{\gamma_t}\frac{\partial f}{\partial t}dv+\int_{\partial\gamma_t}f.\overrightarrow{V}.\overrightarrow{n}ds} \tag{5.19}$$

- $\int_{\gamma_t}\frac{\partial f}{\partial t}dv$: term due to the variation of I with respect to time.
- $\int_{\partial\gamma_t}f.\overrightarrow{V}.\overrightarrow{n}ds$: convection term.

Case of a Discontinuous Function

Assume that $f\left(\overrightarrow{x},t\right)$ is discontinuous across a surface Σ_t.

Assume that $f\left(\overrightarrow{x},t\right)$ is discontinuous across a surface Σ_t with normal vector $\overrightarrow{N}$ oriented from side (1) to side (2) and moving with velocity $\overrightarrow{W}$, see Figure 5.5.

The discontinuity of f across Σ_t is denoted by

---

[2] Let g be a continuous function over a domain occupying the volume γ_t. Let $\overrightarrow{n}$ denote the outward normal vector to $\partial\gamma_t$. Then the following theorem holds: $\int_{\gamma_t}div\left(g\right)dv=\int_{\partial\gamma_t}g.\overrightarrow{n}ds$.

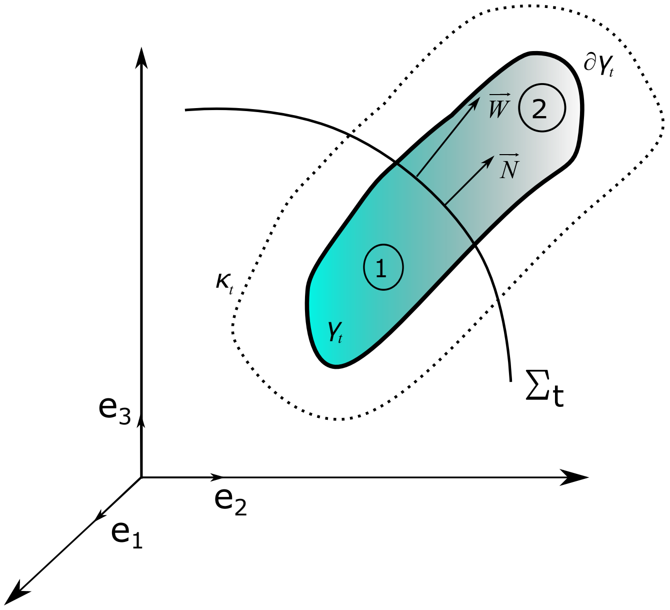


Figure 5.5: Presence of a discontinuity.

$$\|f\| = f_2\left(\overrightarrow{x},t\right) - f_1\left(\overrightarrow{x},t\right)$$

The integral I may be decomposed as

$$I = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \textstyle\int_{\gamma_1} f\left(\overrightarrow{x},t\right) dv \\ I_2 &= \textstyle\int_{\gamma_2} f\left(\overrightarrow{x},t\right) dv \end{aligned}$$

so that

$$\boxed{\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}} \tag{5.20}$$

It is therefore sufficient to compute separately $\frac{dI_1}{dt}$ and $\frac{dI_2}{dt}$ by using Formula (5.19):

$$\frac{dI_1}{dt} = \int_{\gamma_1} \frac{\partial f}{\partial t} dv + \int_{\partial\gamma_1 \cap \partial\gamma_t} f.\overrightarrow{V}.\overrightarrow{n}\, ds + \int_{\Sigma_t} f_1.\overrightarrow{W}.\overrightarrow{N}\, d\Sigma$$

$$\frac{dI_2}{dt} = \int_{\gamma_2} \frac{\partial f}{\partial t} dv + \int_{\partial\gamma_2 \cap \partial\gamma_t} f.\overrightarrow{V}.\overrightarrow{n}\, ds + \int_{\Sigma_t} f_2.\overrightarrow{W}.\left(-\overrightarrow{N}\right) d\Sigma$$

Equation (5.20) then gives

$$\frac{dI}{dt} = \int_{\gamma_t} \frac{\partial f}{\partial t} dv + \int_{\partial\gamma_t} f.\overrightarrow{V}.\overrightarrow{n}\, ds - \int_{\Sigma_t} [|f|].\overrightarrow{W}.\overrightarrow{N}\, d\Sigma$$

Using the generalized divergence theorem,[3] one has

---

[3] Let g be a function defined over a domain occupying the volume γ_t and discontinuous across a surface Σ_t with normal vector $\overrightarrow{N}$. Let $\overrightarrow{n}$ denote the outward normal to $\partial\gamma_t$. Then $\int\limits_{\gamma_t} div g\, dv = \int\limits_{\partial\gamma_t} g.\overrightarrow{n}\, ds - \int_{\Sigma_t} [|g|].\overrightarrow{N}\, d\Sigma$.

$$\int_{\partial\gamma_t} f.\vec{V}.\vec{n}\,ds = \int_{\gamma_t} div\left(f.\vec{V}\right)dv + \int_{\Sigma_t}\left[\left|f\vec{V}\right|\right].\vec{N}\,d\Sigma$$

One finally obtains

$$\boxed{\frac{dI}{dt} = \int_{\gamma_t}\left[\frac{\partial f}{\partial t} + div\left(f.\vec{V}\right)\right]dv + \int_{\Sigma_t}\left[\left|f\left(\vec{V}-\vec{W}\right)\right|\right].\vec{N}\,d\Sigma} \tag{5.21}$$

$\left(\vec{V}-\vec{W}\right)$ represents the relative velocity of the particle with respect to the discontinuity surface Σ_t.

R The surface Σ is not a particle (or material) surface, but rather a geometric surface.

5.6 Exercise

5.6.1 Statement

Consider a motion defined in the basis $B = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ by its Lagrangian representation (ω is a positive constant):

$$\begin{cases} x_1 = X_1.\cos(\omega.t) - X_2.\sin(\omega.t) \\ x_2 = X_1.\sin(\omega.t) + X_2.\cos(\omega.t) \\ x_3 = X_3 \end{cases}$$

1. Compute the gradient tensor $\overline{\overline{F}}$, the stretch tensor $\overline{\overline{C}}$, and the strain tensor $\overline{\overline{E}}$ of this motion at the point $\vec{X}$ and at time t.
2. To which particular class does this motion belong?
3. For a given instant t, compute the stretch at a point X and in a direction $\overrightarrow{dX}$.
4. For a given instant t, compute the shear at a point X and for two directions $\overrightarrow{dX}$ and $\overrightarrow{dX'}$.
5. Consider a medium undergoing this motion, with a homogeneous mass density ρ_0 at the instant $t_0 = 0$. Compute the Jacobian of the transformation and the mass density of the medium at time t.
6. Compute the velocity field $\vec{V}\left(\vec{X},t\right)$ and the acceleration field $\vec{\gamma}\left(\vec{X},t\right)$ in Lagrangian coordinates.
7. Compute the velocity field $\vec{V}(\vec{x},t)$ and the acceleration field $\vec{\gamma}(\vec{x},t)$ in Eulerian coordinates.
8. Compute the Eulerian strain-rate tensor $\overline{\overline{D}}(\vec{x},t)$ and the rotation-rate tensor $\overline{\overline{\Omega}}(\vec{x},t)$.

5.6.2 Solution

- Compute the gradient tensor $\overline{\overline{F}}$, the stretch tensor $\overline{\overline{C}}$, and the strain tensor $\overline{\overline{E}}$ of this motion at the point $\vec{X}$ and at time t.

The gradient tensor is defined by $F_{ij} = \frac{\partial x_i}{\partial X_j}$, hence

$$\overline{\overline{F}} = \begin{bmatrix} \cos(\omega.t) & -\sin(\omega.t) & 0 \\ \sin(\omega.t) & \cos(\omega.t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the stretch tensor, one has $\overline{\overline{C}} = \overline{\overline{F}}^T . \overline{\overline{F}}$, hence

$$\overline{\overline{C}} = \begin{bmatrix} \cos^2(\omega.t) + \sin^2(\omega.t) & \cos(\omega.t).\sin(\omega.t) - \cos(\omega.t).\sin(\omega.t) & 0 \\ \cos(\omega.t).\sin(\omega.t) - \cos(\omega.t).\sin(\omega.t) & \cos^2(\omega.t) + \sin^2(\omega.t) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \overline{\overline{I}}$$

Finally, since $\overline{\overline{E}} = \frac{1}{2}\left(\overline{\overline{C}} - \overline{\overline{I}}\right)$, one has $\overline{\overline{E}} = \overline{\overline{0}}$.

- Since the strain tensor is zero, this is a rigid-body motion.
- Since $\overline{\overline{C}} = \overline{\overline{I}}$, any direction is a principal direction. The stretch in any direction $\overrightarrow{dX}$ is therefore

$$\lambda\left(\overrightarrow{dX}\right) = \sqrt{C_{ii}} = 1$$

- For the same reason, the shear between any two orthogonal directions $\overrightarrow{dX}$ and $\overrightarrow{dX'}$ is

$$\gamma\left(\overrightarrow{dX}, \overrightarrow{dX'}\right) = \frac{C_{ij}}{\sqrt{C_{ii}}.\sqrt{C_{jj}}} = 0$$

- The Jacobian of the transformation is the determinant of $\overline{\overline{F}}$, hence

$$J = \det\left(\overline{\overline{F}}\right) = \begin{vmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\omega t + \sin^2\omega t = 1$$

Consequently, the mass density of the medium remains constant in time and at every point.

- The velocity field $\overrightarrow{V}\left(\overrightarrow{X}, t\right)$ is obtained by differentiation:

$$\overrightarrow{V}\left(\overrightarrow{X}, t\right) = \frac{d\overrightarrow{x}\left(\overrightarrow{X}, t\right)}{dt}$$

$$\begin{cases} x_1 = X_1.\cos(\omega.t) - X_2.\sin(\omega.t) \\ x_2 = X_1.\sin(\omega.t) + X_2.\cos(\omega.t) \\ x_3 = X_3 \end{cases}$$

$$\begin{cases} V_1 = \omega.(-X_1.\sin(\omega.t) - X_2.\cos(\omega.t)) \\ V_2 = \omega.(X_1.\cos(\omega.t) - X_2.\sin(\omega.t)) \\ V_3 = 0 \end{cases}$$

- Similarly, the acceleration field is obtained by differentiation:

$$\overrightarrow{\gamma}\left(\overrightarrow{X}, t\right) = \frac{d\overrightarrow{V}\left(\overrightarrow{X}, t\right)}{dt}$$

$$\begin{cases} \gamma_1 = \omega^2.(-X_1.\cos(\omega.t) + X_2.\sin(\omega.t)) \\ \gamma_2 = \omega^2.(-X_1.\sin(\omega.t) - X_2.\cos(\omega.t)) \\ \gamma_3 = 0 \end{cases}$$

- To compute the velocity field $\overrightarrow{V}(\overrightarrow{x}, t)$ in Eulerian coordinates, one must first determine the function $\overrightarrow{X} = f(\overrightarrow{x}, t)$.

$$x_1.\cos(\omega.t)+x_2.\sin(\omega.t)=X_1.\cos^2(\omega.t)+X_1.\sin^2(\omega.t)=X_1$$

$$-x_1.\sin(\omega.t)+x_2.\cos(\omega.t)=X_2.\sin^2(\omega.t)+X_2.\cos^2(\omega.t)=X_2$$

Substituting this function into the expression of the velocity field in Lagrangian coordinates, one gets

$$\left\{\begin{array}{c} V_1=\omega.\left(-\left(x_1.\cos(\omega.t)+x_2.\sin(\omega.t)\right).\sin(\omega.t)-\left(-x_1.\sin(\omega.t)+x_2.\cos(\omega.t)\right).\cos(\omega.t)\right) \\ V_2=\omega.\left(\left(x_1.\cos(\omega.t)+x_2.\sin(\omega.t)\right).\cos(\omega.t)-\left(-x_1.\sin(\omega.t)+x_2.\cos(\omega.t)\right).\sin(\omega.t)\right) \\ V_3=0 \end{array}\right.$$

- The Eulerian velocity field $\overrightarrow{V}(\overrightarrow{x},t)$ is therefore

$$\left\{\begin{array}{c} V_1=-\omega.x_2 \\ V_2=\omega.x_1 \\ V_3=0 \end{array}\right.$$

The same procedure is applied to the acceleration:

$$\left\{\begin{array}{c} \gamma_1=\omega^2.\left(-\left(x_1.\cos(\omega.t)+x_2.\sin(\omega.t)\right).\cos(\omega.t)+\left(-x_1.\sin(\omega.t)+x_2.\cos(\omega.t)\right).\sin(\omega.t)\right) \\ \gamma_2=\omega^2.\left(-\left(x_1.\cos(\omega.t)+x_2.\sin(\omega.t)\right).\sin(\omega.t)-\left(-x_1.\sin(\omega.t)+x_2.\cos(\omega.t)\right).\cos(\omega.t)\right) \\ \gamma_3=0 \end{array}\right.$$

- The Eulerian acceleration field $\overrightarrow{\gamma}(\overrightarrow{x},t)$ is therefore

$$\left\{\begin{array}{c} \gamma_1=-\omega^2.x_1 \\ \gamma_2=-\omega^2.x_2 \\ \gamma_3=0 \end{array}\right.$$

- Compute the Eulerian strain-rate tensor $\overline{\overline{D}}(\overrightarrow{x},t)$ and the rotation-rate tensor $\overline{\overline{\Omega}}(\overrightarrow{x},t)$.

$$\overline{\overline{D}}=\frac{1}{2}\left(\overline{\overline{L}}+\overline{\overline{L}}^T\right)$$

with $\overline{\overline{L}}=\overline{\overline{grad}}\overrightarrow{V}$ and $\left(\overline{\overline{grad}}\overrightarrow{V}\right)_{ij}=\frac{\partial V_i}{\partial x_j}$.

$$\overline{\overline{L}}=\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The Eulerian strain-rate tensor is

$$\overline{\overline{D}}=\overline{\overline{0}}$$

$$\overline{\overline{\Omega}}=\frac{1}{2}\left(\overline{\overline{L}}-\overline{\overline{L}}^T\right)$$

- The Eulerian rotation-rate tensor is

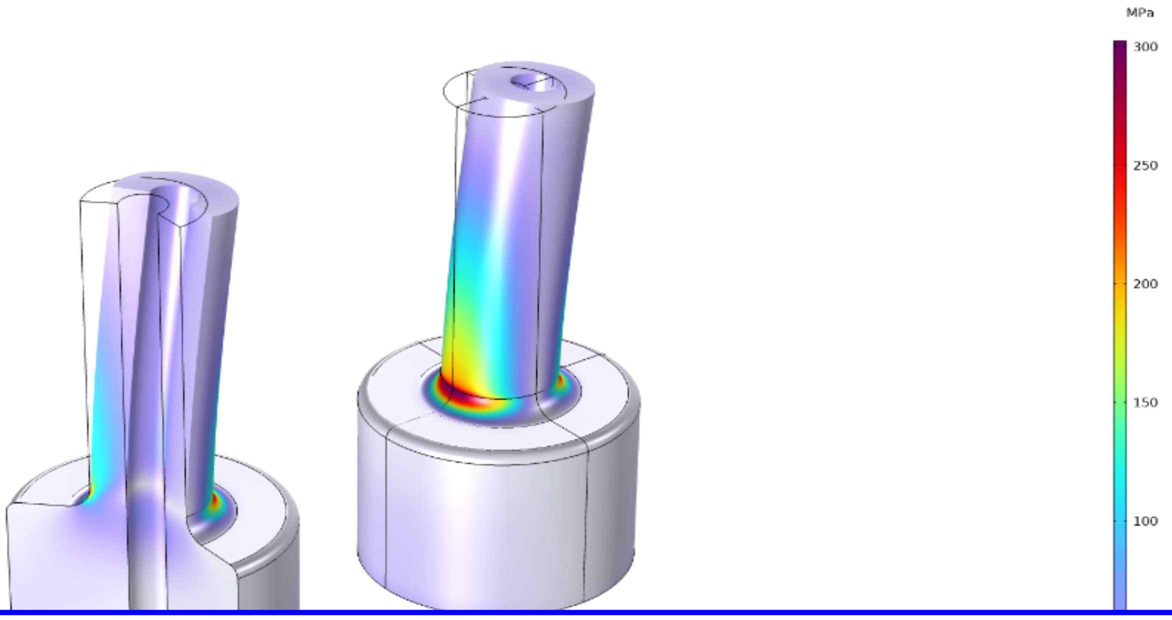
$$\overline{\overline{\Omega}}=\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

III

EFFORTS DANS LES MILIEUX CONTINUS

6 Conservation of Mass: Continuity Equation 91
6.1 Introduction
6.2 First Case
6.3 Second Case

7 Stress 95
7.1 Objective
7.2 Interactions – Newtonian Framework
7.3 Stress Field
7.4 Properties of Stress
7.5 Diagonalization of the Stress Tensor
7.6 Graphical Representation of a Stress State
7.7 Boundary Conditions of the Solid Expressed in Terms of Stress
7.8 Exercise



6. Conservation of Mass: Continuity Equation

6.1 Introduction

In this chapter, we present a first application of the notion of material derivative. The goal is to establish the equation expressing the conservation of mass[1].

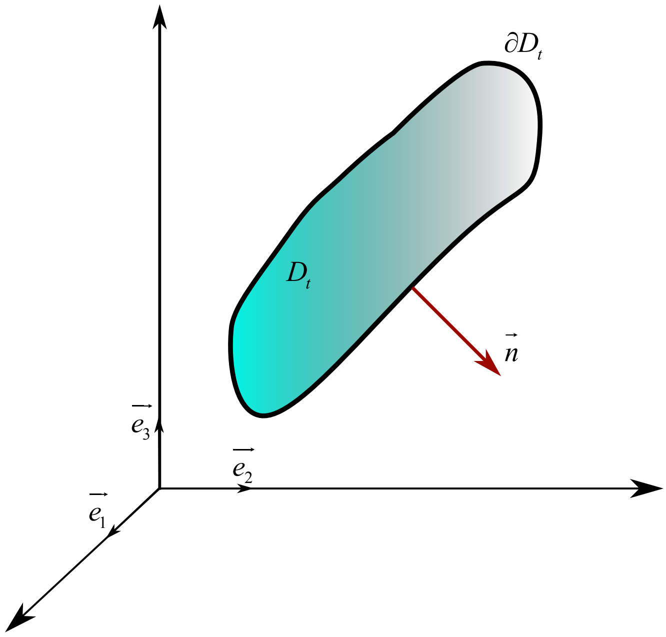


Figure 6.1: Mass of a continuum medium.

Let a medium occupy, at time t, a domain D_t with boundary ∂D_t, outward normal vector $\overrightarrow{n}$, and volume γ_t, see Figure 9.3.

Its volumetric density, or mass density, is denoted by $\rho\left(\overrightarrow{x},t\right)$ and satisfies

---

[1] In classical mechanics, mass is conserved over time, which is not the case in relativistic mechanics.

$$\boxed{\rho\left(\vec{x},0\right)=\rho_0\left(\vec{X}\right)} \tag{6.1}$$

The mass of the volume γ_t is then given by

$$\boxed{M\left(\vec{x},t\right)=\int\limits_{\gamma_t}\rho\left(\vec{x},t\right)dv=\int\limits_{\gamma_0}\rho\left(\vec{\phi}\left(\vec{X},t\right),t\right)Jdv_0} \tag{6.2}$$

The principle of conservation of mass implies that, for **every** material system, the material derivative of the mass is zero:

$$\boxed{\frac{dM}{dt}=0} \tag{6.3}$$

6.2 First Case

ρ and $\vec{V}$ are continuous and continuously differentiable

According to the results of the previous chapter, one has

$$\begin{aligned}\tfrac{dM}{dt}&=\int\limits_{\gamma_t}\tfrac{\partial\rho}{\partial t}dv+\int\limits_{\partial\gamma_t}\rho.\vec{V}.\vec{n}\,ds\\&=\int\limits_{\gamma_t}\tfrac{\partial\rho}{\partial t}dv+\int\limits_{\gamma_t}div\left(\rho.\vec{V}\right).dv\\&=\int\limits_{\gamma_t}\left[\tfrac{\partial\rho}{\partial t}+div\left(\rho.\vec{V}\right)\right]dv\end{aligned} \tag{6.4}$$

Mass conservation is therefore expressed by

$$\forall\gamma_t,\int\limits_{\gamma_t}\left[\frac{\partial\rho}{\partial t}+div\left(\rho.\vec{V}\right)\right]dv=0$$

Hence the local equation of mass conservation:

$$\forall\vec{x}\in D_t,\frac{\partial\rho}{\partial t}+div\left(\rho.\vec{V}\right)=0$$

Now, one has

$$div\left(\rho.\vec{V}\right)=\overrightarrow{grad}\left(\rho\right).\vec{V}+\rho\,div\left(\vec{V}\right)$$

Thus, the mass conservation equation, also called the **continuity equation**, is finally obtained as

$$\boxed{\forall\vec{x}\in D_t,\frac{\partial\rho}{\partial t}+\overrightarrow{grad}(\rho)\cdot\vec{V}+\rho\,div\left(\vec{V}\right)=0} \tag{6.5}$$

Let dm be an elementary mass element. Let dv_0 denote its volume in the reference configuration and dv its volume in the current configuration. Since the mass dm is conserved, one may write

$$dm = \rho_0 dv_0 = \rho dv \Leftrightarrow \frac{\rho}{\rho_0} = \frac{dv_0}{dv}$$

Now, according to Equation (4.12), one has

$$\frac{dv_0}{dv} = \frac{1}{J}$$

hence

$$\boxed{\frac{\rho}{\rho_0} = \frac{1}{J}} \tag{6.6}$$

6.3 Second Case

ρ and $\overrightarrow{V}$ are continuous and continuously differentiable by parts

Let Σ_t be a geometric surface with normal vector $\overrightarrow{N}$, across which ρ and/or $\overrightarrow{V}$ are discontinuous, and let $\overrightarrow{W}$ be the propagation velocity of this surface, see Figure 6.2.

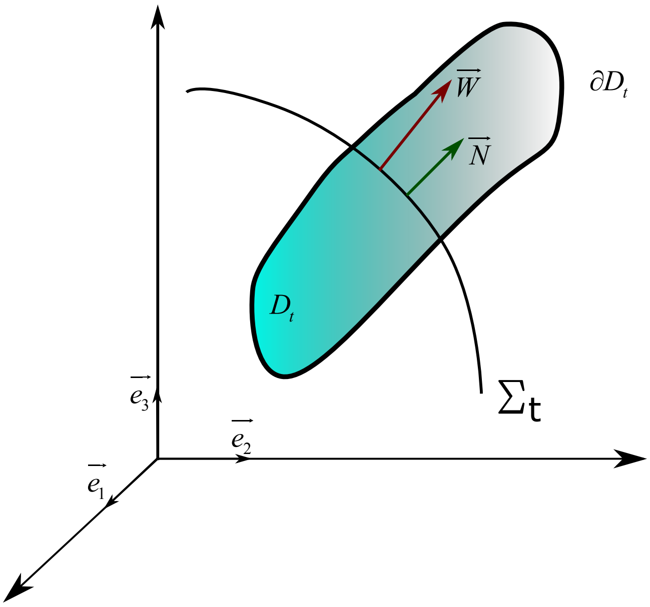


Figure 6.2: Presence of a discontinuity.

Mass conservation is then expressed from Equation (5.21) as

$$\boxed{\forall \gamma_t, \forall \Sigma_t, \frac{dM}{dt} = \int_{\gamma_t} \left[\frac{\partial \rho}{\partial t} + div\left(\rho.\overrightarrow{V}\right)\right] dv + \int_{\Sigma_t} \left[\!\left|\rho\left(\overrightarrow{V} - \overrightarrow{W}\right)\right|\!\right].\overrightarrow{N} d\Sigma} \tag{6.7}$$

By using the arbitrary character of γ_t and Σ_t, one obtains the following local equations:

$$\boxed{\forall \overrightarrow{x} \in D_t, \frac{d\rho}{dt} + \rho\, div\left(\overrightarrow{V}\right) = 0} \tag{6.8}$$

$$\boxed{\forall \overrightarrow{x} \in \Sigma_t, \left[\left|\rho\left(\overrightarrow{V} - \overrightarrow{W}\right)\right|\right].\overrightarrow{N} = 0} \tag{6.9}$$

The quantity

$$\mu = \rho\left(\overrightarrow{V} - \overrightarrow{W}\right).\overrightarrow{N}$$

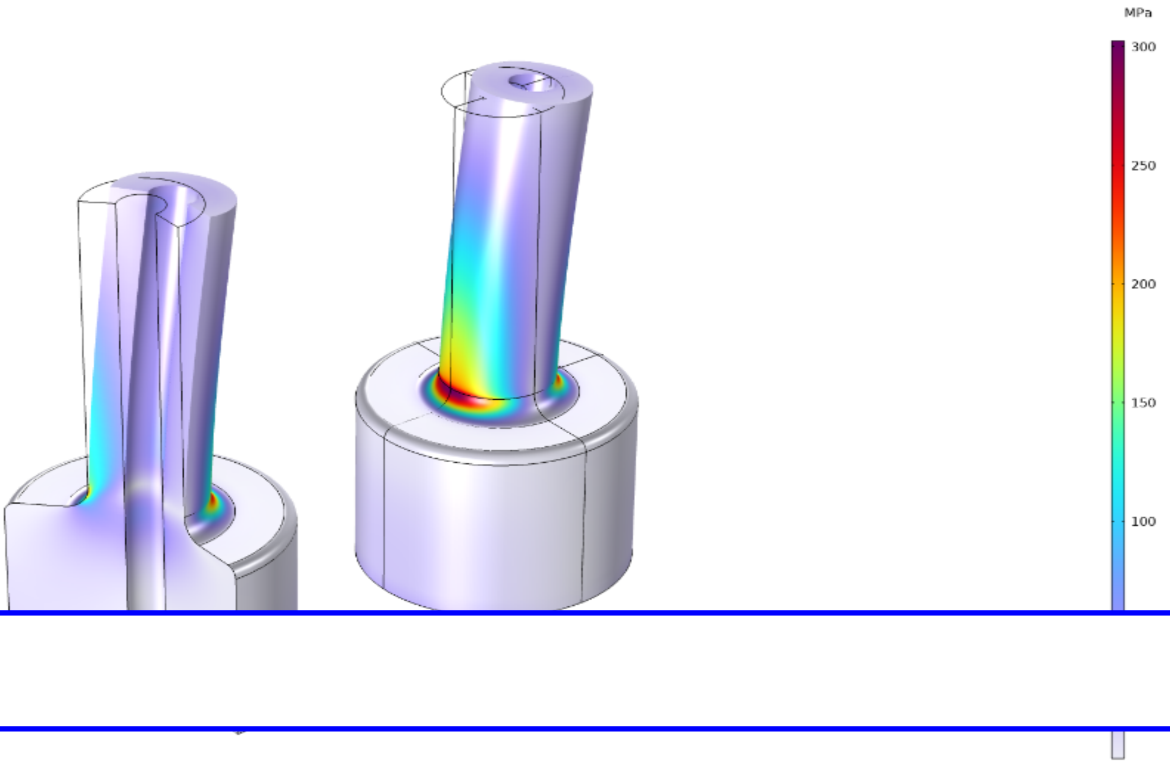
is called the **surface mass flux** through Σ_t. Equation (6.9) may therefore be written as

$$\boxed{\forall \overrightarrow{x} \in \Sigma_t, [|\mu|] = 0} \tag{6.10}$$

which means that the mass flow rate through the geometric surface Σ_t is conserved: “what enters = what leaves,” or in other words, Σ_t does not swallow particles.

R

1. If $\mu \neq 0$, then the particle velocity $\overrightarrow{V}$ is different from the velocity of the discontinuity surface. Hence, Σ_t does not carry the same particles during its motion. It is a geometric surface cutting through the matter. This is called a shock wave.
2. If $\mu = 0$, then $\overrightarrow{V}.\overrightarrow{N} = \overrightarrow{W}.\overrightarrow{N}$, i.e. the normal velocity of the particles is equal to the normal velocity of Σ_t. In this case, it is a particle or material surface. This is called a contact surface. For example, in composite materials, the interface between layers of matter is a contact surface.



7. Stress

7.1 Objective

In this chapter, we are interested in the “internal forces” that develop in a solid subjected to external actions. We shall discuss how these forces are expressed in terms of “stresses,” how stresses are described by a tensor, and how local equilibrium is governed.

In fact, the notion of force, although it seems intuitive at first sight, raises a definition problem. A force is nothing less than a concept of vectorial nature; it is often defined by reference to its ability to accelerate a mass, a thoroughly Newtonian concept.

7.2 Interactions – Newtonian Framework

Although the theory of relativity (Einstein, 1905) denies the uniqueness of time by questioning the notion of simultaneity, the continuum mechanics course adopts the hypotheses formulated by Isaac Newton (1642–1727) on space and time, which prevail in classical mechanics.

7.2.1 Space

Newton assumes a space whose Euclidean structure is independent of the presence of material bodies: “Absolute space, in its own nature, without relation to anything external, remains always similar and immovable.”

Frames moving with uniform rectilinear motion relative to one another are said to be Galilean. In practice, a frame attached to “fixed” (distant) stars of the Galaxy is considered a Galilean frame.

For most applications, especially in civil engineering, a frame attached to the Earth is regarded as a good approximation of a Galilean frame.

7.2.2 Time

Newton assumes an absolute chronology, common to all locations in space: *“Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and is called duration.”*

For him, all clocks are synchronizable regardless of their mutual distance or relative velocity. Therefore, the simultaneity of two events can always be established.

7.2.3 Origin of Forces

Forces between solids (or between different parts of the same solid) mainly result from two interactions between the constituent particles of these solids, among the four known types of interaction (strong interaction, electromagnetic interaction, weak interaction, and gravitational interaction). At the quantum level, these interactions are conveyed by particles called interaction bosons.

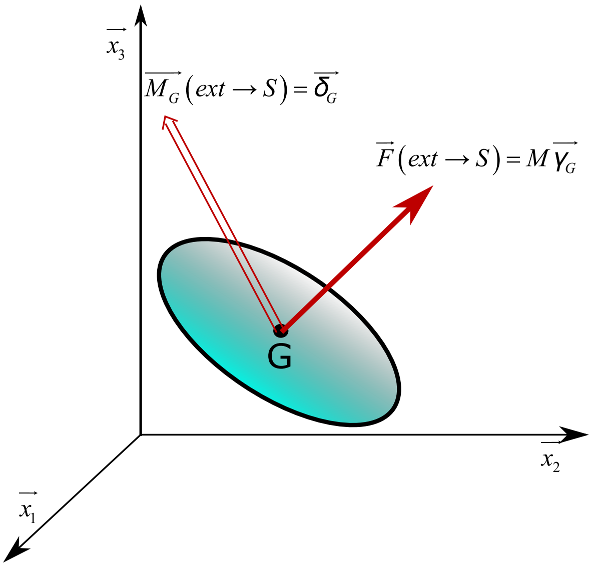


Figure 7.1: Fundamental theorem.

In classical mechanics, only two types of interactions are considered:

1. contact forces: repulsive and of electromagnetic origin;
2. body forces: exclusively attractive and of gravitational origin.

Forces have a vector nature. More precisely, a system of forces may be reduced, from a mathematical point of view, to a wrench.

7.2.4 Fundamental Principle

Dynamics

The fundamental principle of dynamics (or statics in the absence of acceleration) may be stated as follows:

In any Galilean frame, the wrench of external actions $\{F_{ext \rightarrow S}\}$ acting on a solid (or a portion of a solid) is equal to the dynamic wrench $\{D_{S/R_0}\}$ of that solid (or portion of solid). For example, reducing the wrenches at the center of mass:

$$\boxed{\{F_{ext \rightarrow S}\}_G = \{D_{S/R_0}\}_G \Rightarrow \left\{ \begin{array}{c} \vec{F}\,(ext \rightarrow S) = \int_S \rho\,\vec{\gamma}\,dv = M\overrightarrow{\gamma_G} \\ M_G\,(ext \rightarrow S) = \int_S \overrightarrow{GM} \wedge \rho\,\vec{\gamma}\,dv = \overrightarrow{\delta_G} \end{array} \right.} \tag{7.1}$$

In these equations, ρ denotes the mass density, $\vec{\gamma}$ the Galilean acceleration field, dv a volume element, M the total mass of the solid, and $\overrightarrow{\delta_G}$ the dynamic moment computed at the center of mass G.

Statics

In the absence of acceleration (rest), the equations simplify because the wrench of external forces acting on a solid (or a portion of a solid) is equal to the zero wrench:

$$\boxed{\{F_{ext \to S}\}_G = \{0\} \Rightarrow \begin{cases} \overrightarrow{F}\,(ext \to S) = \overrightarrow{0} \\ M_G\,(ext \to S) = \overrightarrow{0} \end{cases}} \tag{7.2}$$

7.2.5 Action–Reaction

It is useful to consider the action of a solid S_1 on another solid S_2 and, conversely, the action of the solid S_2 on the first solid S_1.

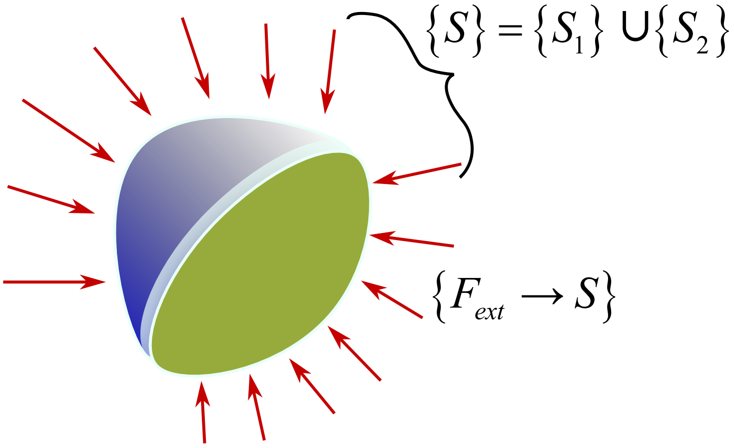


Figure 7.2: Solid considered as the union of its two parts.

- Let $\{F_{ext \to S}\}$ denote the wrench of forces acting on the solid (S) formed by the union of the parts (S_1) and (S_2). These forces act either as body forces (for example, weight) or on the boundary of (S) through its surface. According to the fundamental theorem of dynamics:

$$\boxed{\{F_{ext} \to S\} = \{D_{S/R_0}\}} \tag{7.3}$$

- After separating the two parts, let us focus on part (S_1) and consider the action exerted by part (S_2) on (S_1), reduced to its wrench $\{S_2 \to S_1\}$. Then:

$$\boxed{\{F_{ext} \to S_1\} + \{S_2 \to S_1\} = \{D_{S_1/R_0}\}} \tag{7.4}$$

- Likewise, focusing on part (S_2) and considering the action exerted by part (S_1) on (S_2), reduced to its wrench $\{S_1 \to S_2\}$, one has:

$$\boxed{\{F_{ext} \to S_2\} + \{S_1 \to S_2\} = \{D_{S_2/R_0}\}} \tag{7.5}$$

Adding (7.4) and (7.5) yields:

$$\boxed{[\{F_{ext} \to S_1\} + \{F_{ext} \to S_2\}] + \{S_2 \to S_1\} + \{S_1 \to S_2\} = \left[\{D_{S_1/R_0}\} + \{D_{S_2/R_0}\}\right]} \tag{7.6}$$

Grouping the terms in brackets:

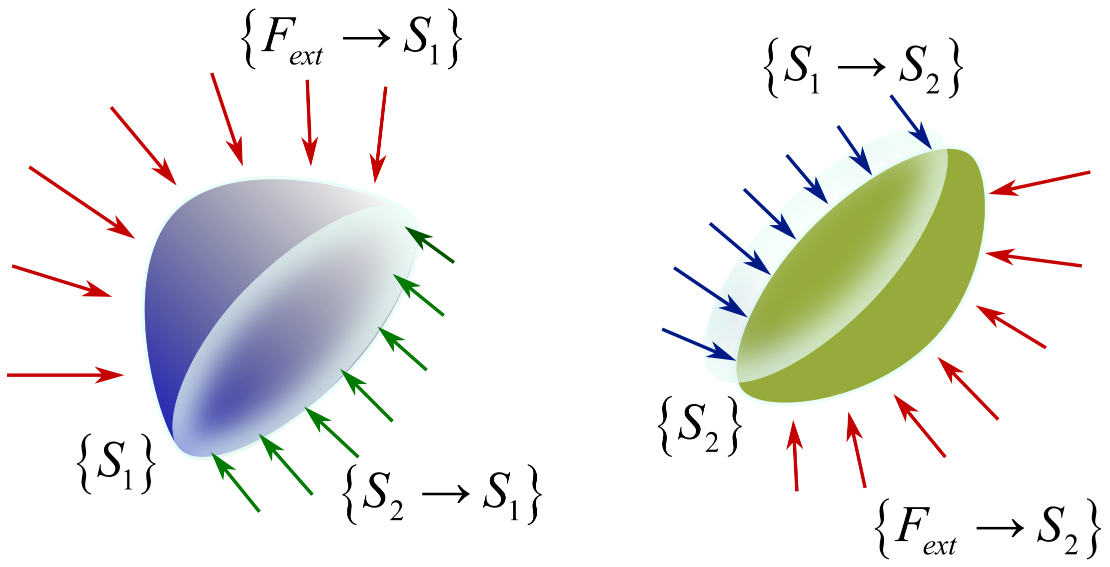


Figure 7.3: Solid decomposed into two parts.

$$\{F_{ext} \rightarrow S\} + \{S_2 \rightarrow S_1\} + \{S_1 \rightarrow S_2\} = \{D_{S/R_0}\}$$

and taking (7.3) into account:

$$\{S_2 \rightarrow S_1\} = -\{S_1 \rightarrow S_2\}$$

The action–reaction principle may thus be stated as follows: the wrench of the actions exerted by the solid (S_2) on the solid (S_1) is equal to the opposite of the wrench of the actions exerted by the solid (S_1) on the solid (S_2).

This principle also holds locally: at every point located on the interface between the two parts, the surface force exerted by the solid S_1 on the solid S_2 is opposite to the surface force exerted by the solid S_2 on the solid S_1.

7.3 Stress Field

7.3.1 Partitioning of a Solid

Let a solid (S) (figure) be subjected to external loads (whether in a dynamic or static regime is irrelevant here).

The solid (S) is partitioned by an arbitrary plane (Figure 7.4). Ignoring the part (S_2), the portion (S_1) is subjected to:

- part of the external loads (body force due to weight + forces applied on the boundary of (S) common to (S_1)),
- all the actions exerted by (S_2) (the “missing” part) on (S_1) through the contact surface (Ω), which express the cohesion between the two portions of the solid when (S) is whole.

7.3.2 Hypothesis on the Nature of Internal Forces

The action of (S_2) on (S_1) is modeled by a distribution of wrenches acting at each point of the cutting surface:

- these wrenches have a resultant homogeneous to a force per unit surface and a moment homogeneous to a couple per unit surface. The distribution of surface moments is only taken

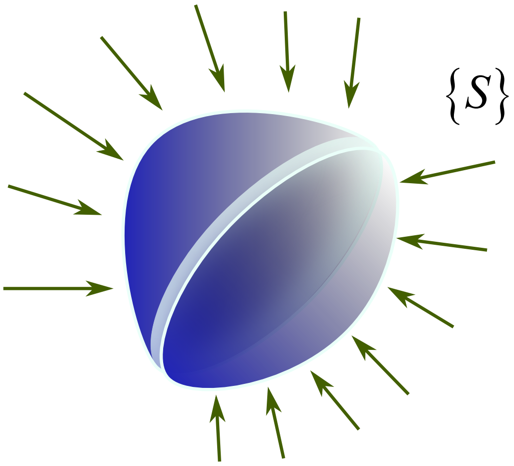


Figure 7.4: Solid subjected to forces.

into account in magnetodynamics, which describes long-range effects caused by a magnetic field (this will not be considered here);

- these wrenches (thus reduced to sliders) depend on the point M at which they are applied;
- these wrenches also depend on the local orientation of the surface (characterized by the outward unit normal vector to (S_1), denoted by $\vec{n}$, as shown in Figure 7.5);

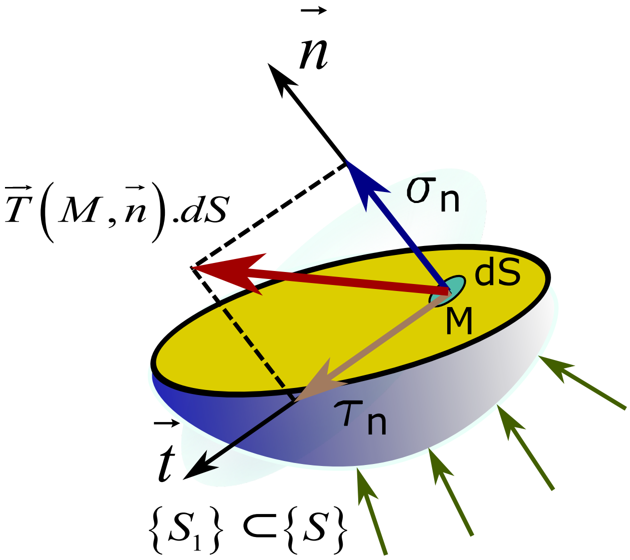


Figure 7.5: Section of the solid.

Cauchy postulate: these wrenches are local in nature (they depend only on M and on $\vec{n}$, and not on the shape or volume of (S_1)).

In ordinary situations, the distribution of surface moments is ignored. We shall therefore make the following assumption:

- the internal contact forces exerted by (S_2) on (S_1) through the common surface are modeled solely by a distribution of surface forces (or force per unit area) depending on the point M of that surface and on the local orientation $\vec{n}$. Dimension of contact surface forces: $\left[F.L^{-2}\right]$, SI unit: Pascal $\left(1Pa = 1N/m^2\right)$, common unit: MegaPascal $\left(1MPa = 10^6 Pa\right)$.
- from a formal mathematical point of view, there therefore exists a mapping, provisionally denoted $\tilde{\sigma}$, which associates a stress vector with each pair formed by a point and a unit

vector:

$$\tilde{\sigma} : (M, \vec{n}) \rightarrow \overrightarrow{T(M, \vec{n})}$$

7.3.3 Normal Stress and Shear Stress

As shown in Figure 7.5, the stress vector $\overrightarrow{T(M, \vec{n})}$ may be projected onto the direction orienting the facet itself and onto the tangent plane to the facet. One may thus write:

$$\overrightarrow{T(M, \vec{n})} = \sigma_n \vec{n} + \tau_n \vec{t}$$

- the vector $\vec{n}$ is perfectly defined, and the component σ_n is called the "normal stress component" or, more simply, the "normal stress"; it is an algebraic scalar component. If it is positive, it produces a local tensile effect at the surface of the solid; if, on the contrary, it is negative, it represents a local compression.
- the vector $\tau_n \vec{t}$ is defined by difference $\tau_n \vec{t} = \overrightarrow{T(M, \vec{n})} - \sigma_n \vec{n}$. The vector $\vec{t}$ is thus defined up to sign, depending on the sign assigned to the component τ_n. The component τ_n has several equivalent names: it is called "tangential stress," "shear stress," or "sliding stress."

7.4 Properties of Stress

In this paragraph, the tensorial nature of the stress field is established, and local equilibrium is stated.

7.4.1 Stress Tensor

Consider a small tetrahedron inscribed inside a solid (S) and centered at a point M. Three of its four facets are constructed parallel to the coordinate planes of the orthogonal basis $\vec{e_1}, \vec{e_2}, \vec{e_3}$ (Figure 7.6).

Facet 1, for example, has an area dS_1 and is oriented by the vector $-\vec{e_1}$. If $\vec{\Phi_1} = \overrightarrow{T(M, \vec{e_1})}$ denotes the stress vector acting on the facet oriented by the vector $+\vec{e_1}$, then by the action–reaction principle, the stress vector acting on facet 1 is opposite to it and therefore equals $-\vec{\Phi_1}$. The same applies to the other orthogonal faces (labeled 2 and 3).

Projecting the vector $\vec{\Phi_k}$, $k = 1..3$, onto the unit vectors of the basis introduces its components: $\vec{\Phi_k} = \sigma_{k1}\vec{e_1} + \sigma_{k2}\vec{e_2} + \sigma_{k3}\vec{e_3}$. The components σ_{kj} are thus assigned a double index: the first index k refers to the unit vector directing the facet, and the second index j to the projection axis, with $k, j = 1..3$.

The equilibrium (static or dynamic) of the tetrahedral domain is established simply by means of a force balance summarized in the following table.

face	direction vector	area	stress vector
1	$-\vec{e_1}$	$dS_1 = \frac{1}{2}dx_2dx_3$	$-\vec{\Phi_1} = \sigma_{11}\vec{e_1} + \sigma_{12}\vec{e_2} + \sigma_{13}\vec{e_3}$
2	$-\vec{e_2}$	$dS_2 = \frac{1}{2}dx_3dx_1$	$-\vec{\Phi_2} = \sigma_{21}\vec{e_1} + \sigma_{22}\vec{e_2} + \sigma_{23}\vec{e_3}$
3	$-\vec{e_3}$	$dS_3 = \frac{1}{2}dx_1dx_2$	$-\vec{\Phi_3} = \sigma_{31}\vec{e_1} + \sigma_{32}\vec{e_2} + \sigma_{33}\vec{e_3}$
4	$\vec{n} = n_1\vec{e_1} + n_2\vec{e_2} + n_3\vec{e_3}$	dS	$\overrightarrow{T(M, \vec{n})} = T_1\vec{e_1} + T_2\vec{e_2} + T_3\vec{e_3}$
		volume	**body vector**
body force due to weight	-	dV	$\rho\vec{g}$
inertial force		dV	$\rho\vec{\gamma}$

Table 7.1: Force balance

With $dS_1 = dS.n_1, dS_2 = dS.n_2, dS_3 = dS.n_3$ (by projection) and $dV = \frac{1}{6}dx_1dx_2dx_3$. Applying the fundamental principle of dynamics to the tetrahedron yields

$$\overrightarrow{T}dS - \overrightarrow{\phi_1}dS_1 - \overrightarrow{\phi_2}dS_2 - \overrightarrow{\phi_3}dS_3 + \rho\overrightarrow{g}dV = \rho\overrightarrow{\gamma}dV$$

That is,

$$\left(\overrightarrow{T} - \overrightarrow{\phi_1}n_1 - \overrightarrow{\phi_2}n_2 - \overrightarrow{\phi_3}n_3\right)dS + \rho\overrightarrow{g}dV = \rho\overrightarrow{\gamma}dV$$

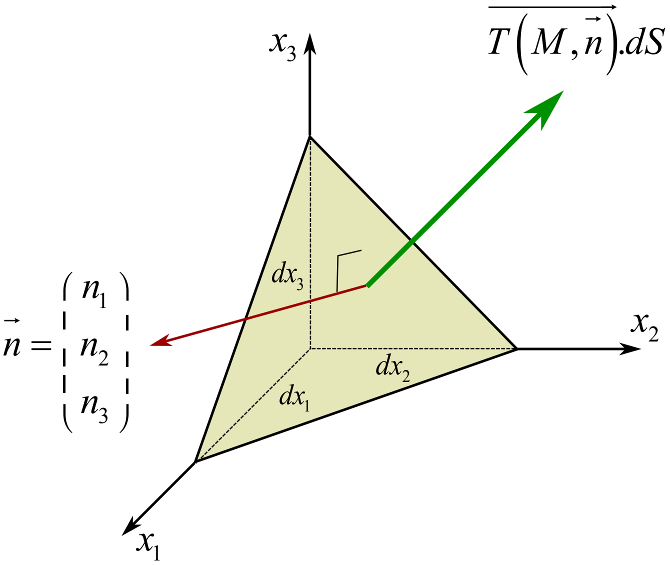


Figure 7.6: Tetrahedral volume element.

When the edges of the tetrahedron tend to zero, the volume terms, being of higher order, become negligible with respect to the surface term, so that

$$\overrightarrow{T} - \overrightarrow{\phi_1}n_1 - \overrightarrow{\phi_2}n_2 - \overrightarrow{\phi_3}n_3 \simeq \overrightarrow{0}$$

and thus

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

We have therefore shown that the operator $\tilde{\sigma} : (M, \overrightarrow{n}) \rightarrow \overrightarrow{T(M, \overrightarrow{n})}$ is linear and of order 2. It is a tensor whose components have just been determined. One writes:

$$\boxed{\overrightarrow{T(M, \overrightarrow{n})} = \overline{\overline{\sigma(M)}}.\overrightarrow{n}} \qquad (7.7)$$

The tensor field $\overline{\overline{\sigma(M)}}$, as a function of the point M, is called the stress field.

7.4.2 Dynamic Equilibrium of a Solid Domain

The distribution of stresses within the solid (S) depends on the loads acting on its surface (∂S) and on the body forces acting on each of its parts. The following derivation aims at establishing the conditions of **local equilibrium**.

R In the literature, one often finds a method consisting in isolating a small rectangular parallelepiped, leading to the same result, although the derivation is then specialized to Cartesian coordinates.

Consider an arbitrary subdomain D extracted from the solid (S) and follow it in its motion (Lagrangian description).

The solid, as a whole, is subjected to external actions applied on its surface (∂S) and to body forces throughout its volume $\{S\}$; under these actions, it undergoes displacements, deformations, and stresses that may vary with time.

Any volume $\{D\}$ extracted from $\{S\}$ is subjected to body forces as well as to the surface forces acting on its boundary (∂D); these surface forces are none other than the stress vectors acting at each point P of the envelope, locally oriented by the outward normal vector $\overrightarrow{n}$ to $\{D\}$.

Dynamic Resultant: Local Equilibrium Equation

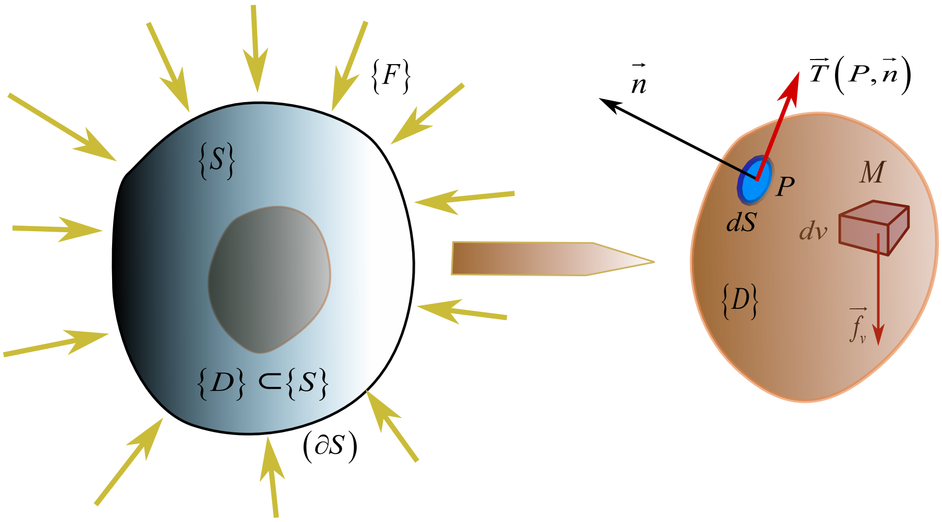


Figure 7.7: Subdomain extracted from a solid.

The resultant form of the fundamental equation of dynamics (7.1) in a Galilean frame R_0 may be written as

$$\iiint_D \overrightarrow{f_v}(M)\,dv + \oint_{\partial D} \overrightarrow{T(P,\overrightarrow{n})}.dS = \iiint_D \rho\,\overrightarrow{\gamma}(M)dv$$

or, taking relation 7.7 into account and adopting simplified notation for the integrals,

$$\int\limits_D \overrightarrow{f_v}(M)\,dv + \oint_{\partial D} \overline{\overline{\sigma(P)}}.\overrightarrow{dS} = \int\limits_D \rho\,\overrightarrow{\gamma}(M)\,dv$$

The divergence theorem (or flux-divergence theorem) makes it possible to transform the surface integral (over the closed boundary) into a volume integral:

$$\int\limits_D \overrightarrow{f_v}(M)\,dv + \int\limits_D \overrightarrow{div}\left(\overline{\overline{\sigma}}\right).dv = \int\limits_D \rho\,\overrightarrow{\gamma}(M)\,dv$$

and since this is true for any $\{D\} \subset \{S\}$, it follows that

$$\boxed{\overrightarrow{div}\left(\overline{\overline{\sigma}}\right)+\overrightarrow{f_v}=\rho\overrightarrow{\gamma}} \tag{7.8}$$

This equation expresses the local equilibrium (in the dynamic sense) of every particle within the solid. In statics, and if one agrees that the body force is nothing but the volumetric weight, one obtains

$$\boxed{\overrightarrow{div}\left(\overline{\overline{\sigma}}\right)+\overrightarrow{f_v}=\overrightarrow{0}} \tag{7.9}$$

Dynamic Moment: Symmetry of the Stress Tensor

The moment form of the fundamental equation of dynamics (7.1) in a Galilean frame R_0, taken with respect to the origin O of the frame, is written as

$$\int_D \overrightarrow{OM}\wedge\overrightarrow{f_v}\left(M\right)dv+\oint_{\partial D}\overrightarrow{OP}\wedge\overrightarrow{T\left(P,\overrightarrow{n}\right)}.dS=\int_D \overrightarrow{OM}\wedge\rho\overrightarrow{\gamma}\left(M\right)dv$$

Using Equation 7.8, the right-hand side may be transformed as

$$\int_D \overrightarrow{OM}\wedge\overrightarrow{f_v}\left(M\right)dv+\oint_{\partial D}\overrightarrow{OP}\wedge\overline{\overline{\sigma}}.\overrightarrow{dS}=\int_D \overrightarrow{OM}\wedge\left(\overrightarrow{div}\overline{\overline{\sigma}}+\overrightarrow{f_v}\right)dv$$

Hence

$$\oint_{\partial D}\overrightarrow{OP}\wedge\overline{\overline{\sigma}}.\overrightarrow{dS}=\int_D \overrightarrow{OM}\wedge\overrightarrow{div}\overline{\overline{\sigma}}dv$$

Let us consider the component along the unit vector $\overrightarrow{e_1}$:

$$\oint_{\partial D}\left(\overrightarrow{OP}\wedge\overline{\overline{\sigma}}\overrightarrow{n}\right).\overrightarrow{e_1}dS=\int_D\left(\overrightarrow{OM}\wedge\overrightarrow{div}\overline{\overline{\sigma}}\right).\overrightarrow{e_1}dv$$

which reduces to

$$\oint_{\partial D}\left(x_2\sigma_{3j}-x_3\sigma_{2j}\right).n_j dS=\int_D\left(x_2\sigma_{3j,j}-x_3\sigma_{2j,j}\right)dv$$

Applying the divergence theorem to the first member gives

$$\int_D \frac{\partial}{\partial x_j}\left(x_2\sigma_{3j}-x_3\sigma_{2j}\right)dv=\int_D\left(x_2\frac{\partial\sigma_{3j}}{\partial x_j}-x_3\frac{\partial\sigma_{2j}}{\partial x_j}\right)dv \quad \forall D$$

Expanding the first term:

$$\delta_{2j}\sigma_{3j}+x_2\frac{\partial\sigma_{3j}}{\partial x_j}-\delta_{3j}\sigma_{2j}-x_3\frac{\partial\sigma_{2j}}{\partial x_j}=x_2\frac{\partial\sigma_{3j}}{\partial x_j}-x_3\frac{\partial\sigma_{2j}}{\partial x_j}$$

$$\Rightarrow \delta_{2j}\sigma_{3j}-\delta_{3j}\sigma_{2j}=0$$

And, since $\delta_{aj}F_{bj}=F_{ab}$, one gets

$$\sigma_{23}-\sigma_{32}=0$$

and, more generally,

$$\sigma_{ij} = \sigma_{ji}$$

The stress tensor is therefore symmetric.

$$\boxed{\overline{\overline{\sigma}} = \overline{\overline{\sigma}}^T} \tag{7.10}$$

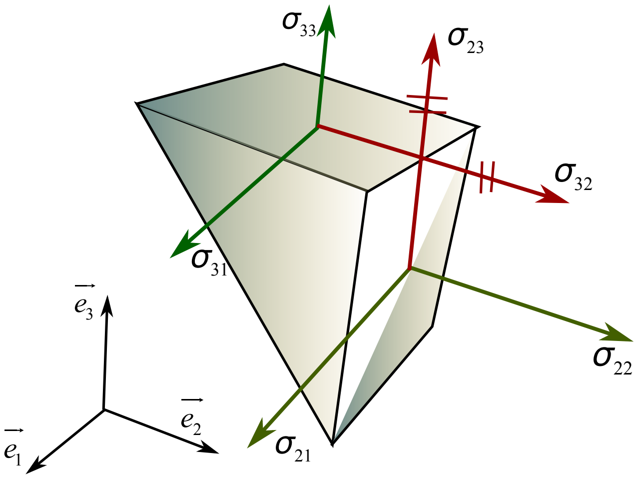


Figure 7.8: Reciprocity of shear stresses.

Interpretation If one considers a right dihedron, and in particular one whose faces are oriented by the vectors $\overrightarrow{e}_i$ and $\overrightarrow{e}_j$, the stress component σ_{ji} acts on the facet oriented by $\overrightarrow{e}_i$ in the direction $\overrightarrow{e}_j$, whereas σ_{ij} acts on the facet oriented by $\overrightarrow{e}_j$ in the direction $\overrightarrow{e}_i$. These two components are equal (for example, the components σ_{23} and σ_{32} in Figure 7.8), which expresses the reciprocity of stresses. This reciprocity guarantees the moment equilibrium of all the particles of the solid.

7.5 Diagonalization of the Stress Tensor

The stress tensor is symmetric. There exists an orthonormal basis in which its representative matrix is diagonal, and the eigenvalues forming its diagonal are real.

7.5.1 Principal Stresses

The principal stresses are the eigenvalues of the stress tensor (Figure 7.9). They are therefore solutions of the characteristic equation: det$\left|\overline{\overline{\sigma}} - \lambda \overline{\overline{I}}\right| = 0$. They are denoted by $\sigma_I, \sigma_{II}, \sigma_{III}$, traditionally ordered from the largest to the smallest.

7.5.2 Principal Stress Directions

The principal stress directions are the eigenvectors forming the orthonormal basis $\overrightarrow{e_I}, \overrightarrow{e_{II}}, \overrightarrow{e_{III}}$, respectively associated with the principal stresses $\sigma_I, \sigma_{II}, \sigma_{III}$.

Thus $\overrightarrow{e}_J$ is a solution of the equation $\left(\overline{\overline{\sigma}} - \sigma_J \overline{\overline{I}}\right).\overrightarrow{e_J} = \overrightarrow{0}$ $(J = I, II, III)$. This homogeneous linear equation admits a vector family as its solution set. One chooses unit vectors forming a direct orthonormal basis.

The stress tensor, in the basis of principal directions, is therefore written as

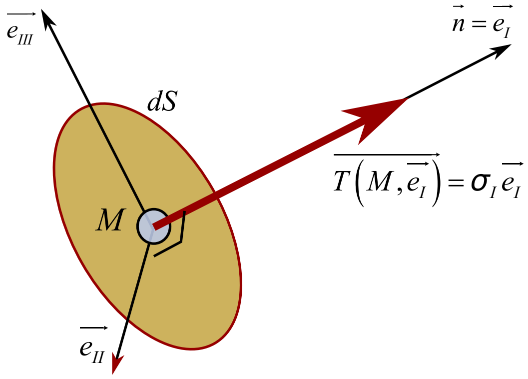


Figure 7.9: Principal stress and principal stress direction.

$$\boxed{\overline{\overline{\sigma}}=\begin{pmatrix}\sigma_I & 0 & 0\\ 0 & \sigma_{II} & 0\\ 0 & 0 & \sigma_{III}\end{pmatrix}_{\{\overrightarrow{e_I},\overrightarrow{e_{II}},\overrightarrow{e_{III}}\}}} \tag{7.11}$$

Property: On a facet oriented by a principal direction vector, there is no shear.

7.5.3 Invariants of the Stress Tensor

The characteristic equation expands as

$$\det\left|\overline{\overline{\sigma}}-\lambda\overline{\overline{I}}\right|=\lambda^3+L_1\lambda^2+L_2\lambda+L_3$$

in which the coefficients are independent of the projection basis of the matrix $\overline{\overline{\sigma}}$: these are the invariants of the stress tensor. Their values are given below:

$$\boxed{\begin{cases} L_1=tr\left(\overline{\overline{\sigma}}\right)\\ L_2=\frac{1}{2}\left(tr^2\left(\overline{\overline{\sigma}}\right)-tr\left(\overline{\overline{\sigma}}^2\right)\right)\\ L_3=\det\left|\overline{\overline{\sigma}}\right|\end{cases}} \tag{7.12}$$

These values may be determined either in an arbitrary basis $\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}$:

$$\begin{cases} L_1=\sigma_{kk}=\sigma_1+\sigma_2+\sigma_3\\ L_2=\frac{1}{2}\left(\sigma_{ii}\sigma_{jj}-\sigma_{ij}\sigma_{ij}\right)\\ L_3=\det\left|\overline{\overline{\sigma}}\right|\end{cases}$$

or in the basis of the principal directions $\overrightarrow{e_I},\overrightarrow{e_{II}},\overrightarrow{e_{III}}$:

$$\begin{cases} L_1=\sigma_{KK}=\sigma_I+\sigma_{II}+\sigma_{III}\\ L_2=\sigma_I\sigma_{II}+\sigma_{II}\sigma_{III}+\sigma_{III}\sigma_I\\ L_3=\sigma_I\sigma_{II}\sigma_{III}\end{cases}$$

7.5.4 Spherical and Deviatoric Parts of a Stress Tensor

Definition

It is likely that immersing a metallic specimen at abyssal depths does not significantly modify its microstructure, the latter already being compact in its natural state. Consequently, the metal is only slightly sensitive to pressure. If one seeks to establish the conditions under which the specimen will be irreversibly affected by a stress state, it is then appropriate not to take into account the part of the stress that may be assimilated to pressure.

For this reason, mechanicians decompose the stress tensor into two terms:

1. a spherical tensor term $\overline{\overline{\sigma_S}}$ containing the entire hydrostatic pressure;
2. a deviatoric tensor term $\overline{\overline{\sigma_D}}$ containing the complementary part of the stress. Since the pressure is entirely contained in the trace of the stress tensor, the deviatoric tensor has zero trace.

Thus, the following relations are established:

$$\overline{\overline{\sigma}} = \overline{\overline{\sigma_S}} + \overline{\overline{\sigma_D}} \quad tr\left(\overline{\overline{\sigma_D}}\right) = 0 \quad tr\left(\overline{\overline{\sigma_S}}\right) = tr\left(\overline{\overline{\sigma}}\right) \quad \overline{\overline{\sigma_S}} = \tfrac{1}{3}\sigma_{kk}\overline{\overline{I}} \quad \overline{\overline{\sigma_D}} = \overline{\overline{\sigma}} - \tfrac{1}{3}tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}}$$

$$\begin{aligned}\overline{\overline{\sigma}} \quad = \quad & \underbrace{\begin{pmatrix} \frac{1}{3}\left(2\sigma_{11} - \sigma_{22} - \sigma_{33}\right) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}\left(2\sigma_{22} - \sigma_{33} - \sigma_{11}\right) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}\left(2\sigma_{33} - \sigma_{11} - \sigma_{22}\right) \end{pmatrix}}_{\overline{\overline{\sigma_D}}} \\ & \underbrace{+\frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\overline{\overline{\sigma_S}}}\end{aligned}$$

The deviator has the same principal directions as the stress tensor.

Invariants of the Deviatoric Tensor

Mechanicians formulate criteria, based for example on the stress tensor but not only on it, that establish the conditions for the onset of irreversible deformations in materials (through plasticity or damage). Naturally, such criteria are expressed independently of the choice of projection basis of tensor quantities and therefore rely on their invariants.

One is thus led to consider the invariants of the deviatoric tensor:

$$\left\{\begin{array}{c} J_1 = tr\left(\overline{\overline{\sigma_D}}\right) = 0 \quad \text{by definition} \\ J_2 = \frac{1}{2}\left(tr\left(\overline{\overline{\sigma_D}}^2\right)\right) \\ J_3 = \det\left|\overline{\overline{\sigma_D}}\right| \end{array}\right.$$

To make these terms more explicit, one may diagonalize the deviator, in the same basis of principal directions as the stress tensor:

$$\overline{\overline{\sigma_D}} = \begin{pmatrix} s_I & 0 & 0 \\ 0 & s_{II} & 0 \\ 0 & 0 & s_{III} \end{pmatrix}_{\{\vec{e_I}, \vec{e_{II}}, \vec{e_{III}}\}}$$

Then

$$\boxed{\left\{\begin{array}{c} J_1 = 0 \\ J_2 = s_I s_{II} + s_{II} s_{III} + s_{III} s_I = \frac{1}{6}\left[\left(\sigma_I - \sigma_{II}\right)^2 + \left(\sigma_{II} - \sigma_{III}\right)^2 + \left(\sigma_{III} - \sigma_I\right)^2\right] \\ J_3 = s_I s_{II} s_{III} \end{array}\right.} \qquad (7.13)$$

7.6 Graphical Representation of a Stress State

7.6.1 MOHR's Three-Circle Diagram

In the basis of the principal stress directions, the representative matrix of the stress tensor at a point M of the solid (S) is given by Equation 7.11. Consider the facet centered at M and oriented by the vector $\overrightarrow{n}$ with components n_1, n_2, n_3 in the basis of eigenvectors $\{\overrightarrow{e_I}, \overrightarrow{e_{II}}, \overrightarrow{e_{III}}\}$. Then, by Equation 7.7, the stress vector acting at M on that facet is

$$\overrightarrow{T\left(M,\overrightarrow{n}\right)} = \overline{\overline{\sigma\left(M\right)}}.\overrightarrow{n} \Rightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \sigma_I n_1 \\ \sigma_{II} n_2 \\ \sigma_{III} n_3 \end{pmatrix}$$

with $\sigma_I > \sigma_{II} > \sigma_{III}$.

One may compute the normal and tangential components of this vector, projected respectively on the facet normal and in its plane:

$$\overrightarrow{T\left(M,\overrightarrow{n}\right)} = \sigma_n \overrightarrow{n} + \tau_n \overrightarrow{t}$$

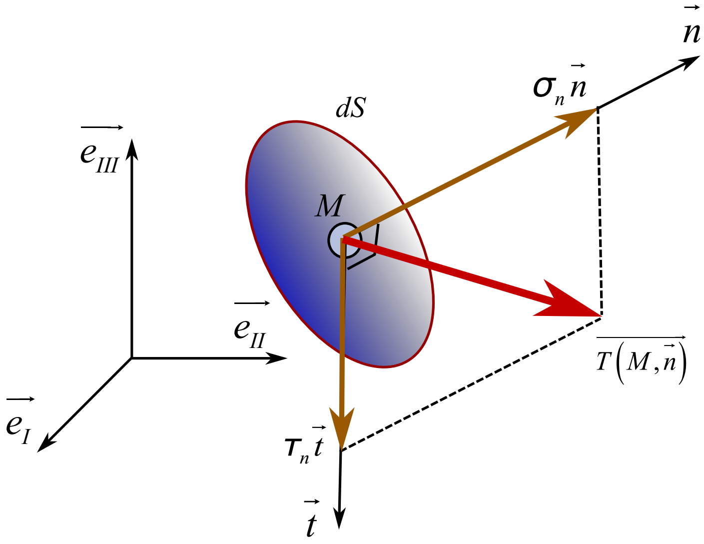


Figure 7.10: Normal stress and shear stress.

Since we are free to choose, let us agree to take τ_n positive, the direction of $\overrightarrow{t}$ following from this choice. Then:

$$\begin{cases} \sigma_n = \overrightarrow{T}.\overrightarrow{n} = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 \\ \tau_n = \left\|\overrightarrow{T} - \sigma_n \overrightarrow{n}\right\| = \sqrt{T^2 - \sigma_n^2} \end{cases}$$

In the end, one gathers the following set of equations and inequalities:

$$\begin{cases} \sigma_n^2 + \tau_n^2 = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2 \\ \sigma_n = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 \\ n_1^2 + n_2^2 + n_3^2 = 1 \\ \sigma_I > \sigma_{II} > \sigma_{III} \end{cases}$$

These equations lead to the following relations:

$$\boxed{\begin{cases} n_1^2 = \frac{\tau_n^2+(\sigma_n-\sigma_{II})(\sigma_n-\sigma_{III})}{(\sigma_I-\sigma_{II})(\sigma_I-\sigma_{III})} \\ n_2^2 = \frac{\tau_n^2+(\sigma_n-\sigma_{III})(\sigma_n-\sigma_{I})}{(\sigma_{II}-\sigma_{III})(\sigma_{II}-\sigma_{I})} \\ n_3^2 = \frac{\tau_n^2+(\sigma_n-\sigma_{I})(\sigma_n-\sigma_{II})}{(\sigma_{III}-\sigma_{I})(\sigma_{III}-\sigma_{II})} \\ \sigma_I > \sigma_{II} > \sigma_{III} \end{cases}} \tag{7.14}$$

Taking into account the inequalities, on the one hand, and the fact that $n_j^2 \geq 0$, on the other hand, one deduces the inequalities

$$\boxed{\begin{cases} \tau_n^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III}) \geq 0 \\ \tau_n^2 + (\sigma_n - \sigma_{III})(\sigma_n - \sigma_{I}) \geq 0 \\ \tau_n^2 + (\sigma_n - \sigma_{I})(\sigma_n - \sigma_{II}) \geq 0 \end{cases}} \tag{7.15}$$

If one associates, to every stress vector $\overrightarrow{T}$, an image point Φ in the plane $\{\sigma, \tau\}$ (MOHR's plane), the previous inequalities show that the point $\Phi(\sigma_n, \tau_n)$ lies in the geometric region bounded by three circles: this is MOHR's three-circle diagram.

7.6.2 MOHR's Circle

A Particular Case of the Stress State

If, locally, the facet (dS) is tangent to one of the three principal planes (containing two principal axes), for example (dS_{III}) tangent to the plane $\{\overrightarrow{e_I}, \overrightarrow{e_{II}}\}$, then the stress vector acting on this facet reduces to the component σ_{III}. This is notably the case when the considered facet lies on the surface of a solid.

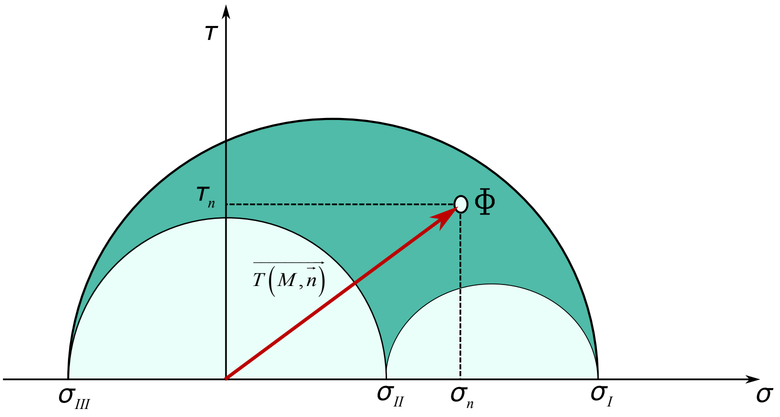


Figure 7.11: MOHR's three-circle diagram (for stresses).

Now consider an arbitrary facet (dS) perpendicular to (dS_{III}), hence containing $\overrightarrow{e_{III}}$ (thus $\overrightarrow{n} \perp \overrightarrow{e_{III}}$ and therefore $n_3 = 0$).

Note that, because of the reciprocity of shear stresses, the shear component $\overrightarrow{\tau_n}$ lies in the plane $\{\overrightarrow{e_I}, \overrightarrow{e_{II}}\}$.

Since $n_3 = 0$, the third equation in the list 7.14 reduces to

$$\tau_n^2 + (\sigma_n - \sigma_I)(\sigma_n - \sigma_{II}) = 0$$

which may be transformed into

$$\left(\sigma_n - \left(\frac{\sigma_I + \sigma_{II}}{2}\right)\right)^2 + \tau_n^2 = \left(\frac{\sigma_I - \sigma_{II}}{2}\right)^2$$

In MOHR's plane, the representative point Φ of the stress vector lies on the circle centered at $\left\{\sigma_0 = \frac{1}{2}\left(\sigma_I + \sigma_{II}\right);0\right\}$ with radius $R = \frac{1}{2}\left(\sigma_I - \sigma_{II}\right)$. The stress state is said to be locally plane; this circle is the MOHR circle of plane stresses. The next paragraph explains its construction.

Construction

The construction of MOHR's circle for stresses concerns a locally plane stress state. The stress state is thus characterized by a stress tensor reduced to

$$\boxed{\overline{\overline{\sigma}} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\{\overrightarrow{e_I},\overrightarrow{e_{II}},\overrightarrow{e_{III}}\}}} \tag{7.16}$$

Conventionally, it is reduced in $2D$ to the form

$$\boxed{\overline{\overline{\tilde{\sigma}}} = \begin{pmatrix} \sigma_I & 0 \\ 0 & \sigma_{II} \end{pmatrix}_{\{\overrightarrow{e_I},\overrightarrow{e_{II}}\}}} \tag{7.17}$$

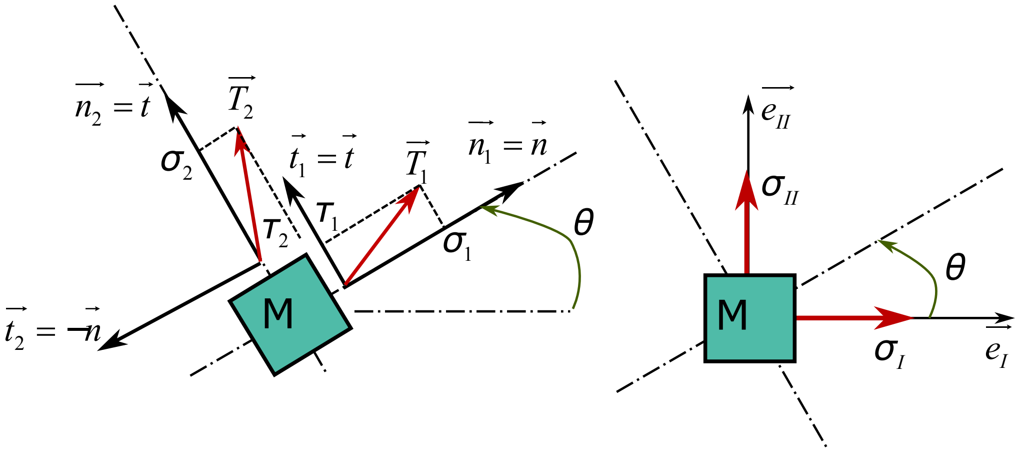


Figure 7.12: Plane stress state (case of the plane perpendicular to axis III).

The purpose of MOHR's circle (Figure 7.12) is to allow the graphical determination, both quantitative and rapid, of the stress vectors acting on facets containing the vector $\overrightarrow{e_{III}}$; therefore, the direction vector of such a facet excludes $\overrightarrow{e_{III}}$ and, consequently,

$$\overrightarrow{n} = n_1\overrightarrow{e_I} + n_2\overrightarrow{e_{II}}$$

n_1 and n_2 are the direction cosines of the vector $\overrightarrow{n}$ in the basis $\overrightarrow{e_I},\overrightarrow{e_{II}}$. Thus one introduces

$$\overrightarrow{n} = \cos\theta\,\overrightarrow{e_I} + \sin\theta\,\overrightarrow{e_{II}}$$

where θ is the angle between the vector $\overrightarrow{n}$ and the axis $\overrightarrow{e_I}$: $\theta = \widehat{e_I, \overrightarrow{n}}$. To simplify the notation further, let $c = \cos\theta$ and $s = \sin\theta$.

The stress vector acting on that facet is therefore

$$\overrightarrow{T_1} = \begin{pmatrix} \sigma_I c \\ \sigma_{II} s \end{pmatrix}$$

in the basis $\{\overrightarrow{e_I}, \overrightarrow{e_{II}}\}$.

This vector is decomposed along the normal $\overrightarrow{n} = \begin{pmatrix} c \\ s \end{pmatrix}$ and the tangent vector $\overrightarrow{t} = \begin{pmatrix} -s \\ c \end{pmatrix}$ obtained by rotating $\overrightarrow{n}$ by $+\frac{\pi}{2}$ about $\overrightarrow{e_{III}}$. Then

$$\overrightarrow{T_1} = \sigma_1 \overrightarrow{n} + \tau_1 \overrightarrow{t} \quad \text{with} \quad \sigma_1 = \sigma_I c^2 + \sigma_{II} s^2 \quad \text{and} \quad \tau_1 = -cs\,(\sigma_I - \sigma_{II})$$

For convenience, introduce the double angle 2θ, so that

$$\begin{array}{r} \sigma_1 = \left(\frac{\sigma_I+\sigma_{II}}{2}\right) + \left(\frac{\sigma_I-\sigma_{II}}{2}\right)\cos 2\theta \\ \tau_1 = -\left(\frac{\sigma_I-\sigma_{II}}{2}\right)\sin 2\theta \end{array}$$

or

$$\sigma_1 = \underbrace{\left(\frac{\sigma_I + \sigma_{II}}{2}\right)}_{\sigma_0} + \underbrace{\left(\frac{\sigma_I - \sigma_{II}}{2}\right)}_{R} \cos(2\theta)$$

$$\tau_1 = -\underbrace{\left(\frac{\sigma_I - \sigma_{II}}{2}\right)}_{R} \sin(2\theta)$$

NOTE 1: In the plane $\{\sigma, \tau\}$, the representative point of the stress vector $\overrightarrow{T_1}$ lies on a circle centered at $\Phi_0\,(\sigma_0, 0)$ and of radius R, with $\sigma_0 = \frac{1}{2}\,(\sigma_I + \sigma_{II})$ and $R = \frac{1}{2}\,(\sigma_I - \sigma_{II})$.

NOTE 2: If the normal to the facet makes an angle θ with the principal direction $\overrightarrow{e_I}$, then the radius $[\Phi_0\Phi]$ makes an angle -2θ with the axis $O\sigma$ (care must be taken with the change in rotation direction).

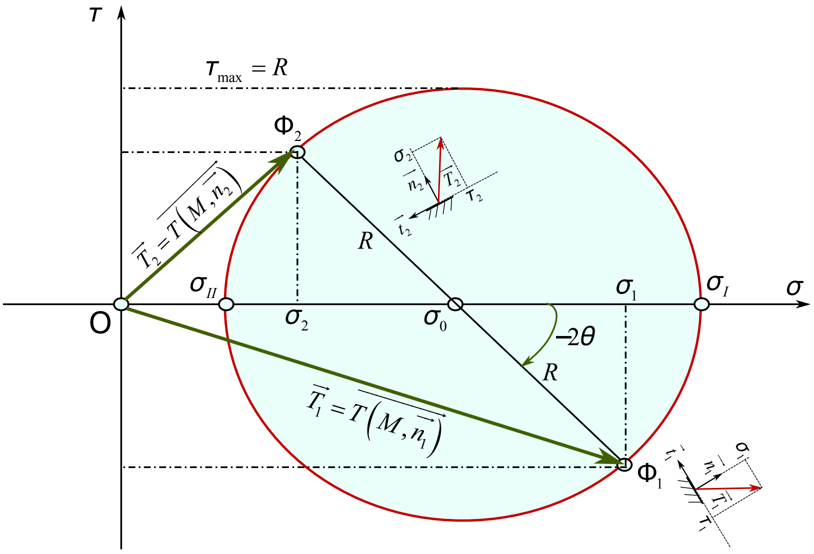


Figure 7.13: MOHR's circle for stresses.

Generalization (Inverse Problem)

When the stress tensor is known in an arbitrary basis:

$$\boxed{\overline{\overline{\sigma}} = \begin{pmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{pmatrix}_{\{\vec{e_1},\vec{e_2}\}}} \tag{7.18}$$

two stress vectors may be readily determined: $\vec{T_1}$ is the one acting on the facet oriented by $\vec{n_1} = \vec{e_1}$, and $\vec{T_2}$ the one acting on the facet oriented by $\vec{n_2} = \vec{e_2}$.

Then

$$\vec{T_1} = \begin{pmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \tau_{12} \end{pmatrix}$$

$$\Rightarrow \vec{T_1} = \sigma_{11}\vec{e_1} + \tau_{12}\vec{e_2} = \sigma_{11}\vec{n_1} + \tau_{12}\vec{t_1}$$

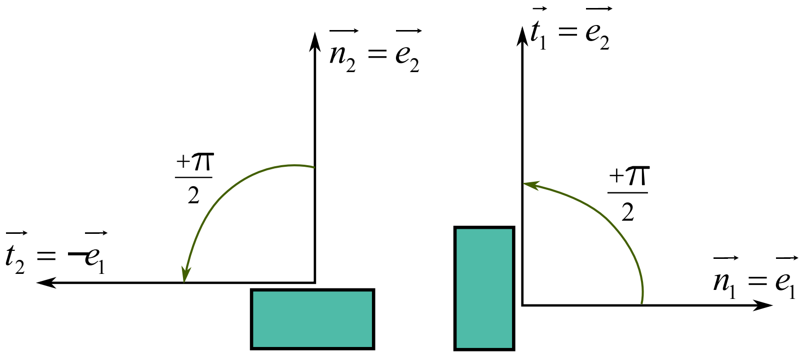


Figure 7.14: Normal and tangent vectors.

and

$$\vec{T_2} = \begin{pmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \tau_{12} \\ \sigma_{22} \end{pmatrix}$$

$$\Rightarrow \vec{T_2} = \tau_{12}\vec{e_1} + \sigma_{22}\vec{e_2} = \sigma_{22}\vec{n_2} - \tau_{12}\vec{t_2} \quad \text{because} \quad \vec{t_2} = -\vec{e_1}$$

Since the rotation angle $\widehat{\vec{n_1},\vec{n_2}}$ is equal to $\frac{\pi}{2}$, the representative points Φ_1 and Φ_2 of the corresponding stress vectors are diametrically opposite, and consequently these two points completely determine MOHR's circle:

$$\begin{cases} \quad center \quad \sigma_0 = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \\ radius \quad \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\tau_{12}^2}/2 \end{cases}$$

From there, the principal stresses are easily deduced:

$$\sigma_I = \sigma_0 + R \quad \sigma_{II} = \sigma_0 - R$$

and then the principal directions follow from

$$\tan(2\theta) = \frac{\tau_{12}}{\sigma_1 - \sigma_0} \Rightarrow \theta = -\frac{1}{2}\arctan\left(\frac{\tau_{12}}{\sigma_1 - \sigma_0}\right)$$

where θ is the inclination angle of the vector $\vec{n_1} = \vec{e_1}$ with respect to the principal direction $\vec{e_I}$.

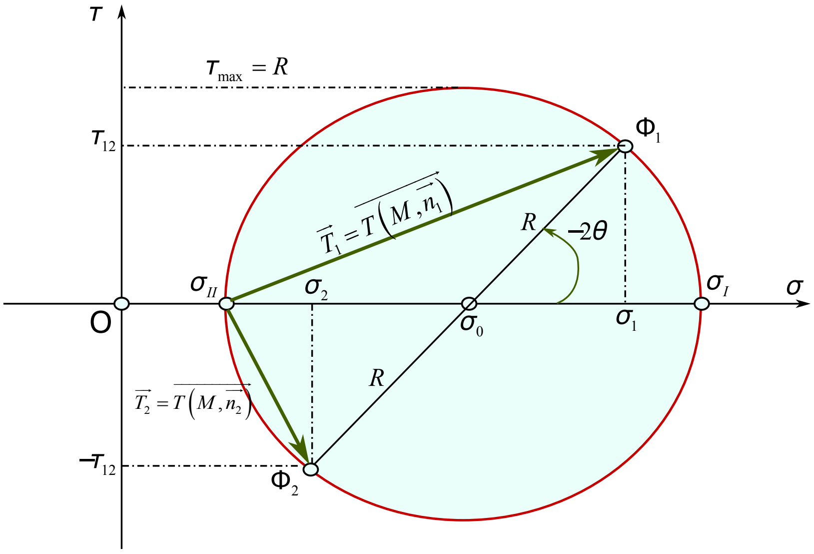


Figure 7.15: Construction of MOHR's circle when the principal stresses are unknown.

7.7 Boundary Conditions of the Solid Expressed in Terms of Stress

On the boundary of the solid, the stress vectors acting on the facets covering the envelope $\{\partial S\}$ are none other than the surface forces applied there.

Where the surface forces are known, that is, on $\{\partial S_f\}$, one writes

$$\boxed{\overrightarrow{T(P)} = \overline{\overline{\sigma}}(P).\overrightarrow{n(P)} = \overrightarrow{f_s(P)} \quad \forall P \in \{\partial S_f\}} \tag{7.19}$$

The steps involved in handling a boundary condition in terms of stress are therefore:

1. determine the equation of the "boundary" (surface in $3D$ or line in $2D$): $\psi(P) = 0$;
2. determine the outward normal vector $\overrightarrow{n(P)}$;
3. compute the stress vector $\overline{\overline{\sigma}}(P).\overrightarrow{n(P)}$;
4. identify it with the surface force: $\overrightarrow{T(P)} = \overrightarrow{f_s(P)}$; one then deduces conditions on the stress tensor.

7.8 Exercise

7.8.1 Statement

At a point M of a continuum medium and at time t, the matrix of Cauchy's stress tensor in a fixed orthonormal Cartesian basis $B = (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ is given by the following expression, where α is a dimensionless constant:

$$\overline{\overline{\sigma}} = \begin{bmatrix} 0.7\alpha & 3.6\alpha & 0 \\ 3.6\alpha & 2.8\alpha & 0 \\ 0 & 0 & 7.6 \end{bmatrix}$$

1. Show that the calculation of the three stresses $\overrightarrow{T}(\overrightarrow{e_i}), i = 1, 2, 3$ and the use of MOHR-circle properties allow one to determine the principal stresses, denoted σ_1, σ_2, and σ_3. Illustrate the case $\alpha = 1$.
2. Determine the values of α corresponding to an axisymmetric triaxial stress state.
3. For the rest of the exercise, let $\alpha = 1$. Determine the principal directions of $\overline{\overline{\sigma}}$ at M.

4. Compute the stress acting on a facet with normal $\overrightarrow{n_1} = \frac{\sqrt{3}}{2}\overrightarrow{e_1} + \frac{1}{2}\overrightarrow{e_2}$. Plot the corresponding point on MOHR's circle.
5. Determine the value of the maximum shear stress and the corresponding normal direction.

7.8.2 Solution

The matrix of the stress tensor in the basis $B = (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ is given as

$$\overline{\overline{\sigma}} = \begin{bmatrix} 0.7\alpha & 3.6\alpha & 0 \\ 3.6\alpha & 2.8\alpha & 0 \\ 0 & 0 & 7.6 \end{bmatrix}$$

As stated in the exercise, let us compute the three stress vectors existing at the point M, for facets whose normals are respectively $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$:

$$\overrightarrow{T}(\overrightarrow{e_1}) = \overline{\overline{\sigma}}\overrightarrow{e_1} = \begin{bmatrix} 0.7\alpha & 3.6\alpha & 0 \\ 3.6\alpha & 2.8\alpha & 0 \\ 0 & 0 & 7.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7\alpha \\ 3.6\alpha \\ 0 \end{bmatrix} = 0.7\alpha\overrightarrow{e_1} + 3.6\alpha\overrightarrow{e_2}$$

$$\overrightarrow{T}(\overrightarrow{e_2}) = \overline{\overline{\sigma}}\overrightarrow{e_2} = \begin{bmatrix} 0.7\alpha & 3.6\alpha & 0 \\ 3.6\alpha & 2.8\alpha & 0 \\ 0 & 0 & 7.6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.6\alpha \\ 2.8\alpha \\ 0 \end{bmatrix} = 3.6\alpha\overrightarrow{e_1} + 2.8\alpha\overrightarrow{e_2}$$

$$\overrightarrow{T}(\overrightarrow{e_3}) = \overline{\overline{\sigma}}\overrightarrow{e_3} = \begin{bmatrix} 0.7\alpha & 3.6\alpha & 0 \\ 3.6\alpha & 2.8\alpha & 0 \\ 0 & 0 & 7.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7.6 \end{bmatrix} = 7.6\overrightarrow{e_3}$$

Each of these vectors may be decomposed into a normal component and a tangential component so as to be represented in MOHR's plane. For example, for the stress vector applied to the facet whose normal is $\overrightarrow{e_1}$, one may write

$$\overrightarrow{T}(\overrightarrow{e_1}) = T_{n1}.\overrightarrow{e_1} + T_{t1}.\overrightarrow{t_1}$$

In this very general equation, $\overrightarrow{t_1}$ is a unit vector belonging to the plane of the facet, and T_{n1} and T_{t1} are respectively the normal and tangential components of the stress vector. By identification with the expression found above, one may write

$$\overrightarrow{t_1} = \overrightarrow{e_2}$$

$$T_{n1} = 0.7\alpha \quad \text{and} \quad T_{t1} = 3.6\alpha$$

Applying the same operation to $\overrightarrow{T}(\overrightarrow{e_2})$ and $\overrightarrow{T}(\overrightarrow{e_3})$, one similarly obtains

$$T_{n2} = 2.8\alpha \quad \text{and} \quad T_{t2} = 3.6\alpha$$

$$T_{n3} = 7.6 \quad \text{and} \quad T_{t3} = 0$$

With these results, one may place in MOHR's plane the points corresponding to each of these three stress vectors. They are the points of coordinates (T_{n1}, T_{t1}), (T_{n2}, T_{t2}), and (T_{n3}, T_{t3}). At this stage, it should be noted that knowledge of these points alone is not sufficient to draw MOHR's three-circle diagram and thus determine the principal stresses.

One must now apply a number of arguments based on properties of MOHR's circle. It is observed that the tangential component of $\overrightarrow{T}(\overrightarrow{e_3})$ is zero, which means by definition that $\overrightarrow{e_3}$ is a principal stress direction. It follows that the vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ lie in the plane of the other two principal directions, and that the points in MOHR's plane corresponding to the stress vectors associated with these facets lie on the same MOHR circle. Consequently, the points (T_{n1}, T_{t1}) and (T_{n2}, T_{t2}) both belong to the same MOHR circle. The same is true for their reflections with respect to the horizontal axis, that is, $(T_{n1}, -T_{t1})$ and $(T_{n2}, -T_{t2})$.

Using the orientation rules given in the course, the two facets with normals $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are orthogonal, and therefore the corresponding points in MOHR's plane must be diametrically opposite on the MOHR circle to which they belong. One deduces that the coordinates of the center of the desired MOHR circle are located at the midpoint of a segment joining two diametrically opposite points:

$$\left(\frac{T_{n1}+T_{n2}}{2}, 0\right) = \left(\frac{0.7\alpha+2.8\alpha}{2}, 0\right) = (1.75\alpha, 0)$$

The radius of this circle is then computed by Pythagoras:

$$R = \sqrt{(2.8\alpha - 1.75\alpha)^2 + (3.6\alpha)^2} = \alpha\sqrt{(2.8-1.75)^2 + (3.6)^2} = 3.75\alpha$$

It follows that one of the principal stresses is equal to $1.75\alpha - 3.75\alpha = -2\alpha$, and another principal stress is $1.75\alpha + 3.75\alpha = 5.5\alpha$. As for the third one, it was already implicitly identified at the beginning of the exercise and is equal to 7.6. The situation is represented in MOHR's plane for $\alpha = 1$:

- An axisymmetric triaxial stress state corresponds to any stress state for which two of the principal stresses are equal.

In the present case, without paying attention to the order of the principal stresses (since that depends on α), one may state that the principal stresses are

$$\sigma_1 = -2\alpha \quad ; \quad \sigma_2 = 5.5\alpha \quad ; \quad \sigma_3 = 7.6$$

There are three situations in which the stress state is axisymmetric triaxial:

$$\sigma_1 = \sigma_2 \Rightarrow -2\alpha = 5.5\alpha \Rightarrow \alpha = 0$$

$$\sigma_1 = \sigma_3 \Rightarrow -2\alpha = 7.6 \Rightarrow \alpha = -3.8$$

$$\sigma_2 = \sigma_3 \Rightarrow 5.5\alpha = 7.6 \Rightarrow \alpha = 1.38$$

- Let $\alpha = 1$. Ordering the stresses gives $\sigma_1 = 7.6$, $\sigma_2 = 5.5$, and $\sigma_3 = -2$.

We seek the principal directions of $\overline{\overline{\sigma}}$ at M.

It has already been shown that $\overrightarrow{e_3}$ is an eigenvector, and when $\alpha = 1$ this eigenvector corresponds to the maximum principal stress. One may therefore state without calculation that

$$\overrightarrow{c_1} = \overrightarrow{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We now seek the coordinates of the other two principal vectors, such that

$$\overrightarrow{c_2} = c_{2,1}.\overrightarrow{e_1} + c_{2,2}.\overrightarrow{e_2}$$

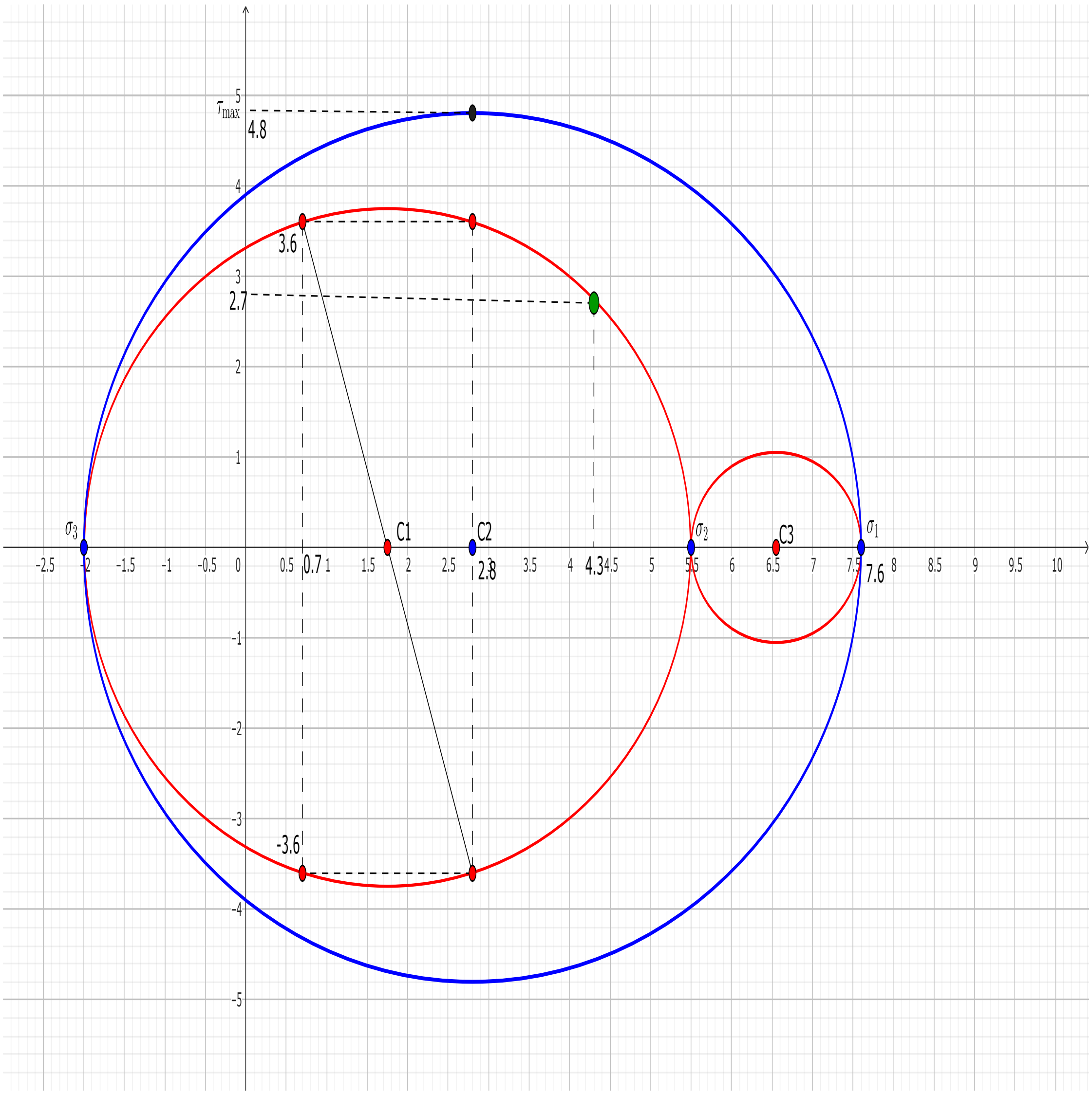


Figure 7.16: Construction of MOHR's circle.

$$\overrightarrow{c_3} = c_{3,1}.\overrightarrow{e_1} + c_{3,2}.\overrightarrow{e_2}$$

Using the classical formula defining a principal stress direction:

$$\overline{\overline{\sigma}}\,\overrightarrow{c_i} = \sigma_i\,\overrightarrow{c_i}$$

One has

$$\overline{\overline{\sigma}}\,\overrightarrow{c_2} = \begin{bmatrix} 0.7 & 3.6 & 0 \\ 3.6 & 2.8 & 0 \\ 0 & 0 & 7.6 \end{bmatrix} \begin{bmatrix} c_{2,1} \\ c_{2,2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7c_{2,1} + 3.6c_{2,2} \\ 3.6c_{2,1} + 2.8c_{2,2} \\ 0 \end{bmatrix}$$

Thus

$$\overline{\overline{\sigma}}\,\overrightarrow{c_2} = (0.7c_{2,1} + 3.6c_{2,2})\,\overrightarrow{e_1} + (3.6c_{2,1} + 2.8c_{2,2})\,\overrightarrow{e_2}$$

On the other hand,

$$\overline{\overline{\sigma}}\overrightarrow{c_2} = \sigma_2\overrightarrow{c_2} = 5.5.\left(c_{2,1}\overrightarrow{e_1} + c_{2,2}\overrightarrow{e_2}\right)$$

Therefore

$$\begin{cases} 5.5c_{2,1} = 0.7c_{2,1} + 3.6c_{2,2} \\ 5.5c_{2,2} = 3.6c_{2,1} + 2.8c_{2,2} \end{cases}$$

$$\begin{cases} -4.8c_{2,1} + 3.6c_{2,2} = 0 \\ 3.6c_{2,1} - 2.7c_{2,2} = 0 \end{cases}$$

The solution set is not unique, so one must add another equation. We use the fact that the vector $\overrightarrow{c_2}$ is unitary. Thus:

$$\begin{cases} -4.8c_{2,1} + 3.6c_{2,2} = 0 \\ c_{2,1}^2 + c_{2,2}^2 = 1 \end{cases}$$

From the first equation, one gets $c_{2,1} = \frac{3}{4}c_{2,2}$, which may be substituted into the second equation:

$$c_{2,2}^2 + \frac{9}{16}c_{2,2}^2 = 1$$

Finally, one obtains

$$\overrightarrow{c_2} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$$

We now use the fact that $(\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3})$ is a direct orthonormal basis. Therefore, the cross product may be used:

$$\overrightarrow{c_3} = \overrightarrow{c_1} \wedge \overrightarrow{c_2}$$

Thus

$$\overrightarrow{c_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix}$$

- Consider the facet whose normal is $\overrightarrow{n_1} = \frac{\sqrt{3}}{2}\overrightarrow{e_1} + \frac{1}{2}\overrightarrow{e_2}$. The stress vector acting on this facet is obtained directly from the course formula:

$$\overrightarrow{T}\left(\overrightarrow{n_1}\right) = \overline{\overline{\sigma}}\overrightarrow{n_1} = \begin{bmatrix} 0.7 & 3.6 & 0 \\ 3.6 & 2.8 & 0 \\ 0 & 0 & 7.6 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 4.5 \\ 0 \end{bmatrix}$$

To plot the corresponding point in MOHR's plane, one needs a normal component and a tangential component. The normal component is obtained, as usual, by projecting $\overrightarrow{T}\left(\overrightarrow{n_1}\right)$ onto the normal direction to the facet:

$$T_n = \overrightarrow{T}\left(\overrightarrow{n_1}\right).\overrightarrow{n_1} = \begin{bmatrix} 2.4 \\ 4.5 \\ 0 \end{bmatrix} . \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{bmatrix} = 4.3$$

The stress vector $\overrightarrow{T}\left(\overrightarrow{n_1}\right)$ is the sum of a vector $T_n\overrightarrow{n_1}$ oriented in the normal direction and another vector oriented in a tangential direction $\overrightarrow{t_1}$ that is not known. The tangential component is therefore the norm of the vector obtained by subtracting the normal component from the stress vector itself:

$$T_t = \left\| \overrightarrow{T}\left(\overrightarrow{n_1}\right) - T_n\overrightarrow{n_1} \right\|$$

$$T_t = \left\| \begin{bmatrix} 2.4 \\ 4.5 \\ 0 \end{bmatrix} - 4.3 \begin{bmatrix} \sqrt{3}/_2 \\ 1/_2 \\ 0 \end{bmatrix} \right\| = 2.7$$

The corresponding point may then be plotted on the graph (see previous figure).

- The maximum shear stress is thus given by

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 4.8$$

The abscissa of the corresponding point in MOHR's plane is that of the center of the largest circle, namely

$$\frac{\sigma_1 + \sigma_3}{2} = 2.8$$

By definition, the facet receiving this stress has a normal belonging to the plane of the two principal directions $\overrightarrow{c_1}$ and $\overrightarrow{c_3}$ (the "plane of maximum shear"). In fact, the facet of maximum shear always corresponds to an orientation located along the bisector of the principal directions $\overrightarrow{c_1}$ and $\overrightarrow{c_3}$, which is therefore given by

$$\frac{\overrightarrow{c_1} + \overrightarrow{c_3}}{2} = \begin{bmatrix} \frac{0-0.8}{2} \\ \frac{0+0.6}{2} \\ \frac{1+0}{2} \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.3 \\ 0.5 \end{bmatrix}$$

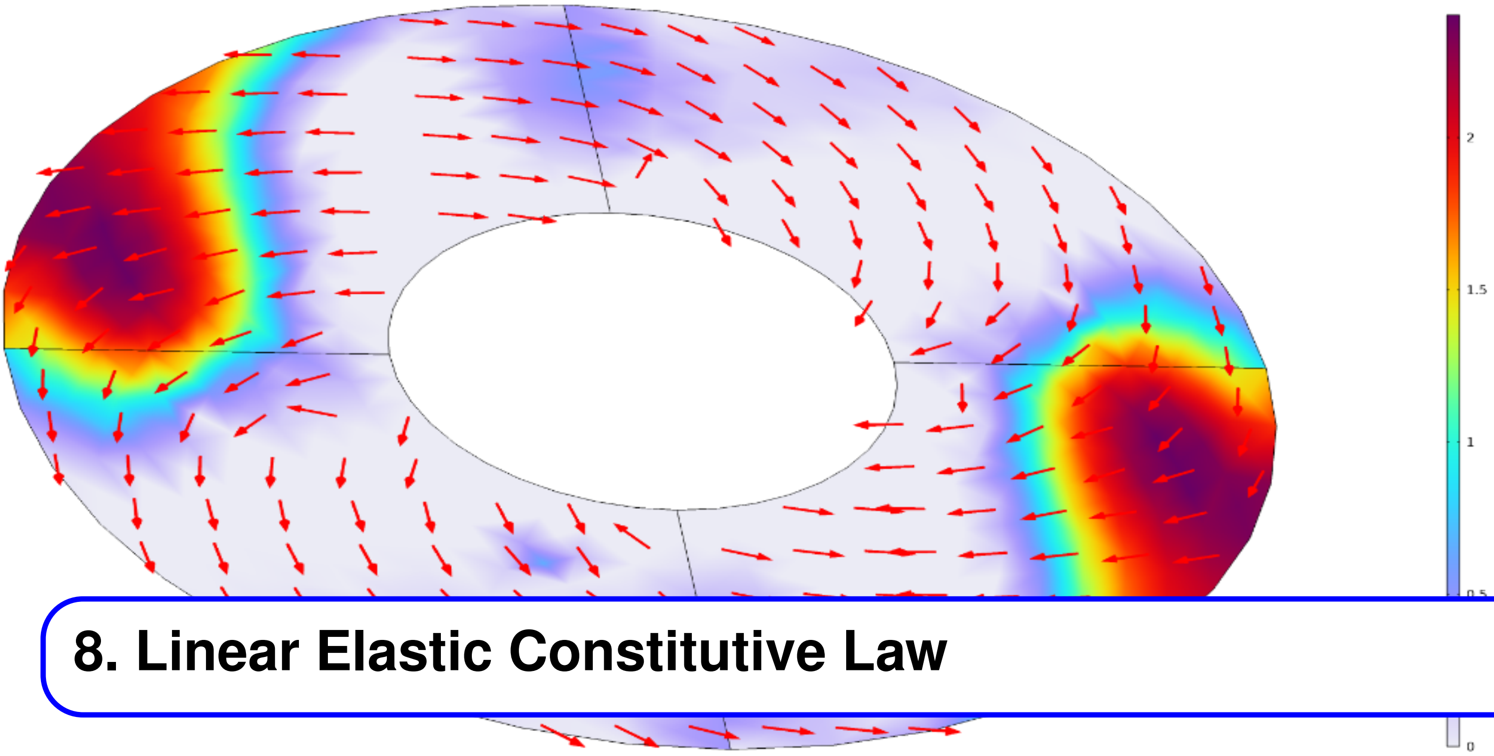
IV

ÉLASTICITÉ

8 Linear Elastic Constitutive Law 121
8.1 Introduction
8.2 Hypotheses
8.3 Formulations of the linear elastic law
8.4 Power, work, energy

9 Methods for Solving Elasticity Problems 131
9.1 Elasticity Problem
9.2 NAVIER Method
9.3 BELTRAMI–MITCHEL Method
9.4 Uniqueness of the Solution of an Elasticity Problem
9.5 RITZ Method
9.6 Two-Dimensional Elasticity
9.7 Axisymmetric Elasticity
9.8 Exercise

Bibliography 153



8. Linear Elastic Constitutive Law

8.1 Introduction

Elasticity theory is a particular field of study of solid materials in which stress and strain are assumed to be related through a "linear" relationship.

This assumption leads to two consequences:

- the effects (strains) are proportional to the causes (stresses); in the case of a uniaxial loading, this amounts to writing $\sigma = E.\varepsilon$. This proportionality can also be extended to the other responses of a solid (or of a structure made up of elastic solids), in particular displacements, which become proportional, or more precisely linearly dependent, on the applied loads.
- deformations are reversible. When the cause disappears, the deformations vanish.

In practice, construction materials obey a wide variety of constitutive laws, often complex.

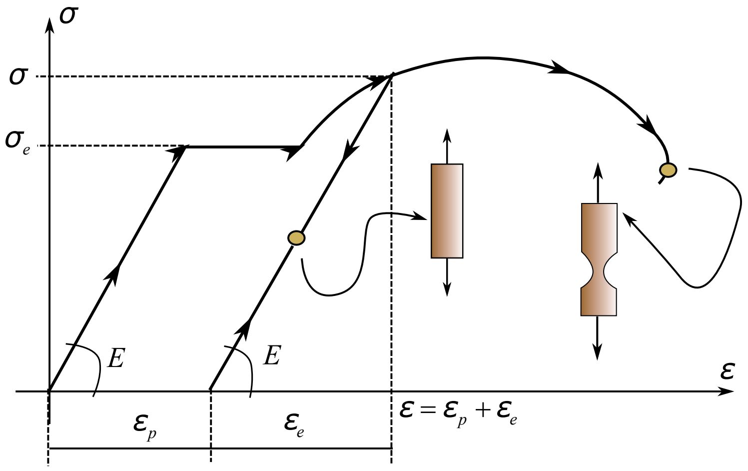


Figure 8.1: Tensile curve.

For example:
steels follow an “elastoplastic” law: if they are deformed beyond a threshold value ε_e, linearity is no longer satisfied. However, if the strain begins to decrease $\dot{\varepsilon} < 0$, linearity is recovered, but expressed in incremental form: $d\sigma = E.d\varepsilon$.

Despite this return to a linear response, irreversibility appears in the form of a residual deformation even when the stress vanishes: this is the plastic strain ε_p.

Concrete and other porous materials undergo “damage” when the stress applied to them exceeds a certain threshold (close to one half of their initial strength). Damage results from the appearance of micro-discontinuities within the material (microcracking, collapse of the porous structure, decohesion of particles), and translates both into a reduction of the slope $E = {}^{d\sigma}\!/_{d\varepsilon}$ and into irreversible strains.

- If it is subjected over a long duration to a stress lower than half of its strength, concrete exhibits the characteristics of a viscoelastic material. This means that deformations increase with time even though the load remains constant. In a uniaxial compression applied at time t_0, the strain at time t is expressed by a creep function $J(t_0,t)$ such that $\varepsilon(t) = J(t_0,t).\sigma_0$

To these must be added other considerations:

- deformations may result from thermal expansion $\varepsilon_{th} = \alpha\Delta T$;
- the cyclic nature of loading gives rise to the phenomenon of “fatigue”, which causes damage even when the stress level remains relatively low with respect to the material strength (figure 8.2);
- the triaxial nature of certain loadings;
- the consolidation capacity of certain materials under hydrostatic pressure;
- the possibility of viscoplasticity;
- the asymmetry of material behavior in tension and compression (concrete);
- the anisotropic nature of certain materials (wood, extruded ceramics, fiber-reinforced composite materials);
- the composite nature of certain solids (reinforced concrete, carbon-based composites);
- the sensitivity of the mechanical properties of materials to temperature (all materials), and even to hygrometry (wood, concrete);
- the evolving, aging nature of materials throughout their history due to the migration of water or chemical species (concrete, glass);
- coupling with physicochemical actions (electromagnetic radiation, adsorption-desorption, chemical attack, corrosion);
- dynamic effects (impacts).

As can be seen, the assumption of elastic material behavior does not account for all the phenomena mentioned above. Nevertheless, it remains an applicable law provided that the stress level is not too high. Engineers therefore rely on this assumption as a first approach and then perform additional a posteriori verifications; thus, in reinforced concrete, for example, it is possible to determine the internal forces using the assumption of elasticity, and to design sections at the serviceability limit states with the same assumption, the European design codes (Eurocodes) specifying the calculation procedures.

Whenever required by the situation (special structures, a need for high accuracy, research), more elaborate models are developed or used through their implementation in numerical computation codes.

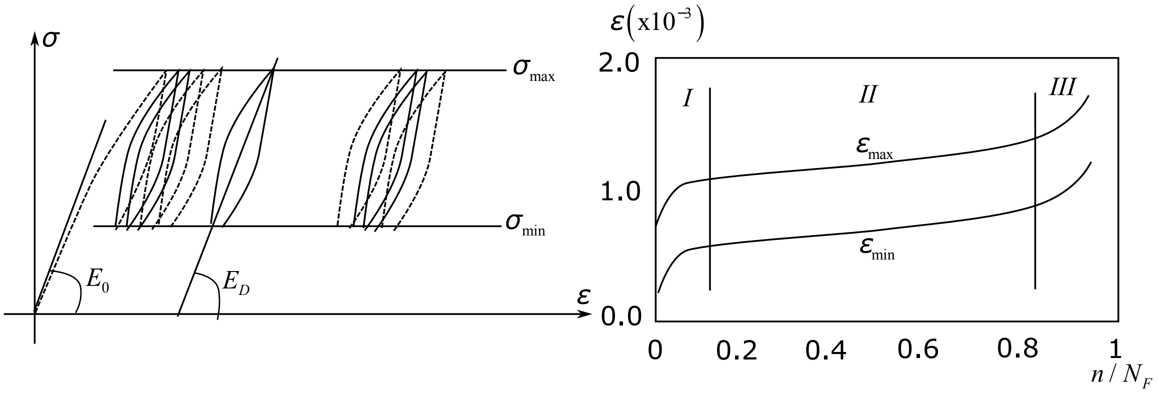


Figure 8.2: Strain–number of cycles curve.

8.2 Hypotheses

8.2.1 Experimentation

Without going into the details of the experiments specifically aimed at "identifying" the mechanical behavior of all materials, let us focus on the characterization of two of them: steel and concrete (with regard to short-duration loading).

8.2.2 Behavior of steel (tensile test)

A profiled cylindrical specimen is placed between the grips of a tensile testing machine. It is equipped with strain gauges allowing the measurement of both the axial strain ε_{11} (change in length) and the transverse strain ε_{22} (change in circumference). It is subjected to an increasing tensile force F. It is assumed that, in the central zone of initial length L, the tensile stress σ_{11} is uniform and equal to $\sigma = F/A$ where A is the area of the initial cross-section.

The test is strain-controlled, and the evolution of the pair $\{\varepsilon_{11}, \sigma_{11}\} = \{\varepsilon, \sigma\}$ is recorded at constant strain rate $\dot{\varepsilon}$. Four stages are observed:

1. a stage during which ε and σ increase proportionally to one another as long as the stress remains lower than a value σ_e called the "elastic limit"; this value is close to $360MPa$ for mild structural steels, for example, and may reach $1\,600MPa$ for the steel used in prestressing cables. During this stage, if the specimen is unloaded, the loading history is found to be totally reversible. When the stress vanishes, no residual strain is observed. This phase is called the "elastic phase". It is characterized by the constant ratio ${}^{\sigma}/_{\varepsilon} = E$ called the "modulus of elasticity". This modulus is close to $200\,000MPa = 200GPa$ for most steels. If the transverse strain is observed, it is found to be negative: while the specimen elongates, its section decreases. Moreover, the ratio ${}^{\varepsilon_{22}}/_{\varepsilon_{11}}$, which is negative, is also constant. One writes $\varepsilon_{22} = -\nu\varepsilon_{11}$ where ν is the contraction coefficient, also called the **Poisson coefficient**. ν is dimensionless and is approximately equal to 0.3 for steels. Thus, in the elastic regime:

$$\boxed{\sigma_{11} = E\varepsilon_{11} \quad \varepsilon_{22} = -\nu\varepsilon_{11}} \tag{8.1}$$

2. a stage during which the strain increases while the stress remains constant. This plateau is called the "plastic plateau" and the material is said to "flow". At the nanoscale, this phase is accompanied by rearrangements within the atomic lattice: dislocations arising from local irregularities propagate. If the strain rate were stopped and then reversed, an immediate decrease in stress would be observed. The ratio $\Delta\sigma/\Delta\varepsilon$ then regains the value it had in the elastic regime; this proves that the material is not damaged. However, a residual strain ε_p remains when the stress becomes zero.

3. if the deformation continues to increase, the stress starts increasing again. Since the dislocations have been exhausted, slip mechanisms occur within the atomic structure. The material is said to "strain harden". If the specimen is unloaded from a stress level $\sigma_s > \sigma_e$, the modulus is found to remain unchanged, while the plastic strain has increased. If the material is reloaded, it first behaves elastically $\left(\frac{d\sigma}{d\varepsilon} = E\right)$, and strain hardening resumes at the stress value σ_s, as if the material had retained this value in memory.
4. as the axial strain continues to increase, the value of σ reaches a maximum, considered to be the "ultimate stress" of the material. This is followed by a decrease in tensile stress accompanied by a localized reduction in the section of the specimen: this is "necking". The test may then be continued until the specimen separates into two pieces.

8.3 Formulations of the linear elastic law

If the loading applied to the solid induces a multiaxial stress field, linear elasticity is expressed by a linear relation between the stress and strain tensors (apart from thermal effects):

$$\boxed{\overline{\overline{\varepsilon}} = \overline{\overline{\overline{\overline{S}}}}\left(\overline{\overline{\sigma}} - \overline{\overline{\sigma_0}}\right) + \overline{\overline{\alpha}}\delta T} \tag{8.2}$$

At the reference temperature T_0, the solid may be the seat of a residual stress field called "self-stress" $\overline{\overline{\sigma_0}}$, even in the absence of external forces.

$\overline{\overline{\overline{\overline{S}}}}$ is the fourth-order tensor of "compliances".

$\overline{\overline{\alpha}}$ is the tensor of thermal expansion coefficients. Expression (8.2) simplifies when the material is isotropic, which will be assumed in the following.

8.3.1 HOOKE–DUHAMEL law

The HOOKE–DUHAMEL law summarizes and generalizes the previous assumptions to the case of multiaxial loading applied to a homogeneous isotropic solid. It expresses the strain tensor as a function of the stress tensor.

Robert Hooke
1635 - 1703

R Robert Hooke, born in Freshwater in 1635 and deceased in London in 1703, was an English polymath scientist. He is regarded as one of the greatest experimental scientists of the 17th century and as one of the key figures of the Scientific Revolution of the early modern period. The British historian of science Allan Chapman referred to him as the "*Leonardo of England*".

Thus:

$$\boxed{\overline{\overline{\varepsilon}} = \frac{(1+\nu)}{E}\overline{\overline{\sigma}} - \frac{\nu}{E}tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} + \underbrace{\alpha\delta T}_{\varepsilon_{th}}\overline{\overline{I}}} \tag{8.3}$$

In this relation appear:

- E, the modulus of elasticity or **Young's modulus**. Its dimension is that of a stress $[E] = \left[F.L^{-2}\right]$ and its SI unit is the Pascal Pa, although in most practical applications the megapascal MPa (or even the gigapascal GPa) is preferred for convenience.

Thomas Young
1773 - 1829

R Thomas Young, born in Milverton in 1773 and deceased in London in 1829, was a British physicist, physician, and Egyptologist. His excellence in many unrelated fields made him a polymath, much like Leonardo da Vinci. His knowledge was so broad that he became known as the Young phenomenon. He practiced medicine throughout his life, but is especially known for defining Young's modulus in materials science.

- ν, the contraction coefficient or **Poisson's ratio**. It is dimensionless and lies in the interval $[0;0.5]$.

Siméon Denis Poisson
1781 - 1840

R Siméon Denis Poisson, born in Pithiviers in 1781 and deceased in Sceaux in 1840, was a French mathematician, geometer, and physicist. His most important contribution concerned electricity and magnetism, fields which he helped establish. He introduced a famous correction to Laplace's second-order equation for the potential, now known as Poisson's equation, published in 1813. It is also in his treatise on mechanics that he introduced the coefficient bearing his name, one of the fundamental quantities of elasticity theory.

- α, the coefficient of thermal expansion. Its dimension is the inverse of a temperature $[\alpha] = \left[T^{-1}\right]$, and its unit is therefore the inverse Kelvin K^{-1}; δT represents the temperature difference with respect to the reference temperature T_0.

This relation shows, importantly, that the principal directions of stress and strain coincide.

8.3.2 GABRIEL LAMÉ law

The LAMÉ–DUHAMEL law is the reciprocal form of the previous one (eq. 8.3), in the sense that this time the stresses are expressed in terms of strain (and temperature):

$$\boxed{\overline{\overline{\sigma}} = 2\mu\overline{\overline{\varepsilon}} + \lambda tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}} - \beta\delta T\overline{\overline{I}}} \qquad (8.4)$$

In this relation:

- μ is the **first Lamé coefficient**. It may be noted that if the stress state is reduced to a shear stress σ_{12}, then $\sigma_{12} = 2\mu\varepsilon_{12}$. Its dimension is that of a shear stress $[\mu] = \left[F.L^{-2}\right]$ and its unit is the Pascal Pa.
- λ is the **second Lamé coefficient**. Its dimension is that of a stress $[\lambda] = \left[F.L^{-2}\right]$ and its unit is the Pascal Pa.
- β is a **thermal stress coefficient**: it is clear that if deformations are prevented (confined material) while the temperature increases, a spherical stress state develops whose component is negative (pressure) $\sigma^{thermal} = -p = -\beta\delta T$. The dimension of β is therefore that of stress divided by temperature: $[\beta] = \left[F.L^{-2}.T^{-1}\right]$. This coefficient is expressed in Pascal per Kelvin, Pa/K.

8.3.3 Relations between coefficients

The equivalence between relations (eq. 8.4) and (eq. 8.3) makes it possible to establish the dependency relations between the various coefficients. Thus, one has:

LAMÉ → HOOKE

$$\boxed{\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu = G = \frac{E}{2(1+\nu)} \\ \beta = \frac{\alpha E}{(1-2\nu)} \\ 2\mu + \lambda = E\frac{1-\nu}{(1+\nu)(1-2\nu)} \\ 3\lambda + 2\mu = \frac{E}{1-2\nu} \end{cases}} \qquad (8.5)$$

HOOKE → LAMÉ

$$\boxed{\begin{cases} E = \mu\frac{3\lambda+2\mu}{\lambda+\mu} \\ \nu = \frac{\lambda}{2(\lambda+\mu)} \\ \alpha = \frac{\beta}{3\lambda+2\mu} \\ \frac{1}{1+\nu} = \frac{2(\lambda+\mu)}{3\lambda+2\mu} \\ \frac{\nu}{1+\nu} = \frac{\lambda}{3\lambda+2\mu} \end{cases}} \qquad (8.6)$$

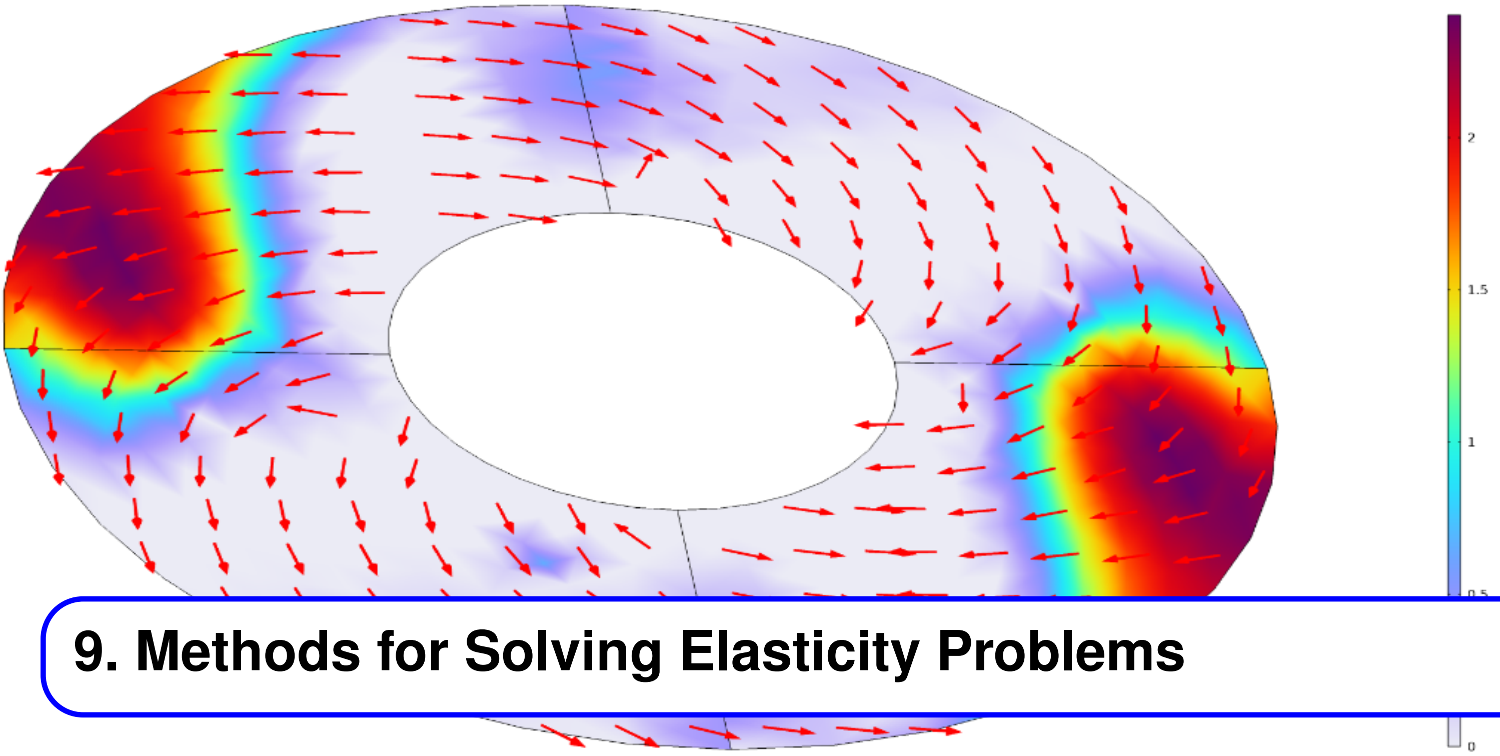
8.3.4 Cross formulations

Combining the previous relations makes it possible to express the stress tensor as a function of the strain tensor and temperature using only Young's modulus, Poisson's ratio, and the coefficient of thermal expansion:

$$\boxed{\overline{\overline{\sigma}} = \frac{E}{1+\nu}\left(\overline{\overline{\varepsilon}} + \frac{\nu}{1-2\nu} tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}}\right) - \frac{\alpha E}{1-2\nu}\Delta T\overline{\overline{I}}} \tag{8.7}$$

And conversely:

$$\boxed{\overline{\overline{\varepsilon}} = \frac{1}{2\mu}\left(\overline{\overline{\sigma}} - \frac{\lambda}{3\lambda+2\mu} tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}}\right) + \frac{\beta}{3\lambda+2\mu}\Delta T\overline{\overline{I}}} \tag{8.8}$$



9. Methods for Solving Elasticity Problems

9.1 Elasticity Problem

9.1.1 Problem statement

Consider a solid $\{S\}$ subjected to body forces $\overrightarrow{f_v(M)}$ and bounded by its surface (∂S). This boundary is partitioned as follows (see Figure 9.1):

- (∂S_u): region where displacements are prescribed (support surface);
- (∂S_f): region where surface tractions $\overrightarrow{f_s(P)}$ are prescribed.

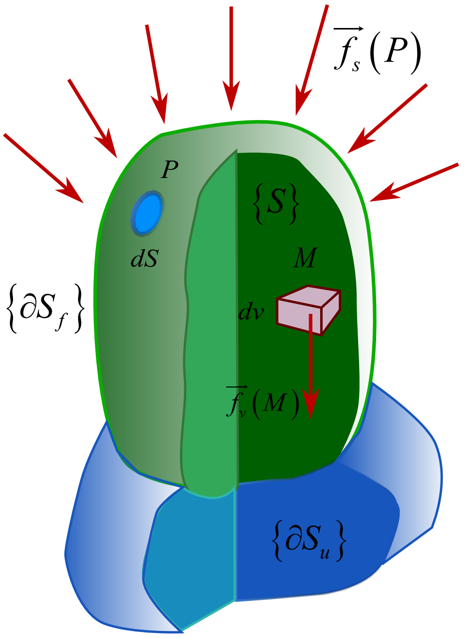


Figure 9.1: Data of an elasticity problem.

We also assume that the following hypotheses hold:

1. continuity hypothesis;
2. small-perturbation hypothesis (SPH): displacements are small compared with the dimensions of the solid, so that, to first approximation, the equilibrium equations may be established in the initial, undeformed configuration of the structure;
3. elastic behavior hypothesis for the solid. In addition, deformations are assumed to be small and reversible, and Hooke's law is assumed valid.

9.1.2 Objective

1. determine the stress field $\overline{\overline{\sigma(M)}}$. The engineer needs to know the level of loading in the solid under various loading conditions. Knowledge of the stress field makes it possible to assess the strength of the solid and the reversibility of deformations.
2. determine the strain field $\overline{\overline{\varepsilon(M)}}$.
3. determine the displacement field $\overrightarrow{u(M)}$ and assess the proper functioning of a mechanical assembly or the framework of a structure.

9.1.3 Summary of the available equations

Equations governing the stress field

Static equilibrium:

$$\boxed{\sigma_{ij} = \sigma_{ji}} \tag{9.1}$$

$$\boxed{\overrightarrow{div}\,\overline{\overline{\sigma}} + \overrightarrow{f_v} = \overrightarrow{0}} \tag{9.2}$$

Boundary conditions:

$$\boxed{\overrightarrow{T(P)} = \overline{\overline{\sigma(P)}}.\overrightarrow{n(P)} = \overrightarrow{f_s(P)} \quad \forall P \in \left\{\partial S_f\right\} \quad \text{and} \quad \vec{n} \perp \{\partial S\} \quad \text{outward}} \tag{9.3}$$

Equations governing the strain and displacement fields

Compatibility equations:

$$\boxed{\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}^T \quad \Leftrightarrow \quad \varepsilon_{ij} = \varepsilon_{ji}} \tag{9.4}$$

$$\boxed{\overline{\overline{grad}}\left(\overrightarrow{div}\left(\overline{\overline{\varepsilon}}\right)\right) + \overline{\overline{grad}}^T\left(\overrightarrow{div}\left(\overline{\overline{\varepsilon}}\right)\right) - \overline{\overline{grad}}\left(\overrightarrow{grad}\left(tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{\varepsilon}}\right)\right) - \Delta\left(\overline{\overline{\varepsilon}}\right) = \overline{\overline{0}}} \tag{9.5}$$

Displacement–strain relation:

$$\boxed{\overline{\overline{\varepsilon}} = \frac{1}{2}\left(\overline{\overline{grad}}\,\vec{u} + \overline{\overline{grad}}^T\,\vec{u}\right)} \tag{9.6}$$

Boundary conditions:

$$\boxed{u_i = u_i^0 \quad on \quad \{\partial S_u\}} \tag{9.7}$$

Constitutive law

The following laws are all equivalent:
HOOKE–DUHAMEL law:

$$\boxed{\overline{\overline{\varepsilon}} = \frac{(1+\nu)}{E}\overline{\overline{\sigma}} - \frac{\nu}{E}tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} + \underbrace{\alpha\delta T}_{\varepsilon_{th}}\overline{\overline{I}}} \tag{9.8}$$

conversely:

$$\boxed{\overline{\overline{\sigma}} = \frac{E}{1+\nu}\left(\overline{\overline{\varepsilon}} + \frac{\nu}{1-2\nu}tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}}\right) - \frac{\alpha E}{1-2\nu}\Delta T\overline{\overline{I}}} \tag{9.9}$$

LAMÉ–DUHAMEL law:

$$\boxed{\overline{\overline{\sigma}} = 2\mu\overline{\overline{\varepsilon}} + \lambda tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}} - \beta\delta T\overline{\overline{I}}} \tag{9.10}$$

and:

$$\boxed{\overline{\overline{\varepsilon}} = \frac{1}{2\mu}\left(\overline{\overline{\sigma}} - \frac{\lambda}{3\lambda+2\mu}tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}}\right) + \frac{\beta}{3\lambda+2\mu}\Delta T\overline{\overline{I}}} \tag{9.11}$$

Overview

Solving an elasticity problem means determining 15 pointwise functions which, in Cartesian coordinates, are: $\{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}, u_1, u_2, u_3, \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}\}$. To do so, one has 3 local equilibrium equations, 6 constitutive relations, and 6 compatibility equations, i.e., 15 equations in total.

The problem is therefore well posed; a priori, an analytical solution should be obtainable for each problem. In practice, however, numerical computation, based on subdividing the solid into small volumetric elements called finite elements, is necessary in most practical situations.

NOTE: The number of static equations is smaller than the number of static unknowns: the equilibrium equations alone generally do not suffice to solve an elasticity problem, unless the number of unknowns is reduced by additional assumptions. A solid, viewed from the standpoint of continuum mechanics, constitutes a statically indeterminate system.

Methodology

To solve an elasticity problem, the engineer must combine the above equations. Before doing so, a primary objective must be chosen: should the problem be approached from the standpoint of internal forces, in which case priority is given to determining the stress field, or is it preferable first to determine the displacement field?

Whatever the primary objective, the remaining unknowns are subsequently deduced by traversing, in one direction or the other, the methodological cycle shown opposite.

Once the objective has been chosen, solving the problem is “reduced” to the search for analytical solutions of a set of rather general partial differential equations discussed in the following sections. These must be supplemented by boundary conditions in order for the particular, unique solution to be fully determined.

If the problem exhibits geometric particularities, these are taken into account as early as possible so as to reduce the mathematical complexity. Among the possible simplifications, note the following situations, which will be developed later:

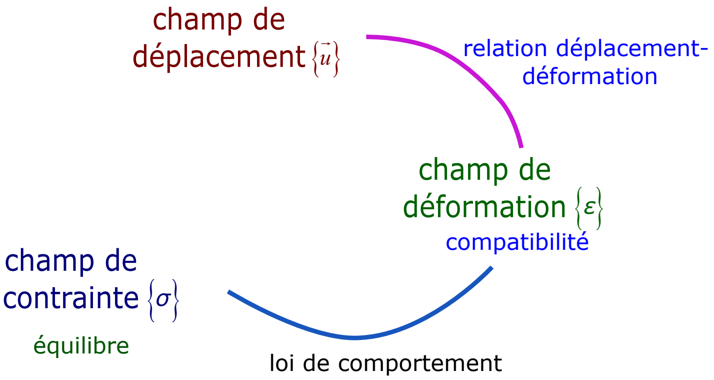


Figure 9.2: Methodology for solving an elasticity problem.

- plane stress state;
- plane strain state;
- axisymmetric problem.

9.2 NAVIER Method

9.2.1 Principle

The Navier method belongs to a "displacement method." It is used whenever the primary objective is the determination of the displacement field. To obtain the differential equations governing the field $\left\{\overrightarrow{u(M)}\right\}$, one successively combines:

- the local equilibrium equation;
- the elastic constitutive law (Lamé form);
- the displacement–strain relation.

9.2.2 Equations

Local equilibrium (9.2):

$$\overrightarrow{div}\,\overline{\overline{\sigma}} + \overrightarrow{f_v} = \overrightarrow{0}$$

Taking into account Lamé's law (9.10) under isothermal conditions:

$$\overrightarrow{div}\left[2\mu\overline{\overline{\varepsilon}} + \lambda tr\left(\overline{\overline{\varepsilon}}\right)\overline{\overline{I}}\right] + \overrightarrow{f_v} = \overrightarrow{0}$$

Then, using the displacement–strain relation (9.6):

$$2\mu\overrightarrow{div}\left[\frac{1}{2}\left(\overline{\overline{grad}}\,\overrightarrow{u} + \overline{\overline{grad}}^{T}\,\overrightarrow{u}\right)\right] + \lambda\overrightarrow{div}\left(div\,\overrightarrow{u}.\overline{\overline{I}}\right) + \overrightarrow{f_v} = \overrightarrow{0}$$

Now, $\overrightarrow{div}\left(w.\overline{\overline{I}}\right) = \overrightarrow{grad}\,(w)$, $\overrightarrow{div}\left(\overline{\overline{grad}}^{T}\,\overrightarrow{u}\right) = \overrightarrow{grad}\,div\,\overrightarrow{u}$, and also: $\overrightarrow{div}\left(\overline{\overline{grad}}\,\overrightarrow{u}\right) = \overline{\overline{\Delta}}\,\overrightarrow{u}$
Hence:

$$\mu\,\overrightarrow{\Delta}\,\overrightarrow{u} + (\mu+\lambda)\,\overrightarrow{grad}\,div\,\overrightarrow{u} + \overrightarrow{f_v} = \overrightarrow{0}$$

Noting that $\overrightarrow{\Delta}\,\overrightarrow{u} = \overrightarrow{grad}\,div\,\overrightarrow{u} - \mu\overrightarrow{rot}\,\overrightarrow{rot}\,\overrightarrow{u}$, one obtains another form of the same equation:

$$(\lambda+2\mu)\,\overrightarrow{grad}\,div\,\overrightarrow{u} - \mu\overrightarrow{rot}\,\overrightarrow{rot}\,\overrightarrow{u} + \overrightarrow{f_v} = \overrightarrow{0}$$

Let us summarize Navier's equations here for reference:

$$\boxed{\mu\,\overrightarrow{\Delta}\,\overrightarrow{u} + (\mu+\lambda)\,\overrightarrow{grad}\,div\,\overrightarrow{u} + \overrightarrow{f_v} = \overrightarrow{0}} \tag{9.12}$$

$$\boxed{(\lambda+2\mu)\,\overrightarrow{grad}\,div\,\overrightarrow{u} - \mu\,\overrightarrow{rot}\,\overrightarrow{rot}\,\overrightarrow{u} + \overrightarrow{f_v} = \overrightarrow{0}} \tag{9.13}$$

To these must be added the displacement boundary conditions already stated in (9.7):

$$u_i = u_i^0 \quad on \quad \{\partial S_u\}$$

9.2.3 Kinematically admissible displacement field

A “kinematically admissible displacement field” (KA field) is any virtual displacement field $\left\{\overrightarrow{u^*}\right\}$ that is continuous and differentiable (at least piecewise) and satisfies the boundary conditions of the solid:

$$u_i^* = u_i^0 \quad on \quad \{\partial S_u\}$$

Such fields are used in approximate methods (for example the Ritz method in finite elements, or limit analysis methods in elastoplastic analysis of metallic structures).

NOTE: The “real” displacement field (exact solution of Navier’s equations) is kinematically admissible.

9.3 BELTRAMI–MITCHEL Method

9.3.1 Principle

Beltrami’s method is a “force method”; it is used when the primary objective is to establish the stress field. To do so, the compatibility equations are written in terms of stresses by means of the elastic constitutive law, while taking local equilibrium into account.

9.3.2 Equations

Let us start from the 6 compatibility equations (9.5):

$$\overline{\overline{grad}}\left(\overrightarrow{div}\left(\overline{\overline{\varepsilon}}\right)\right) + \overline{\overline{grad}}^T\left(\overrightarrow{div}\left(\overline{\overline{\varepsilon}}\right)\right) - \overline{\overline{grad}}\left(\overrightarrow{grad}\left(tr\left(\overline{\overline{\varepsilon}}\right)\right)\right) - \overline{\overline{\Delta}}\left(\overline{\overline{\varepsilon}}\right) = \overline{\overline{0}}$$

Now take into account the constitutive law (9.9):

$$\overline{\overline{\varepsilon}} = \tfrac{(1+\nu)}{E}\overline{\overline{\sigma}} - \tfrac{\nu}{E}tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}} \Rightarrow tr\left(\overline{\overline{\varepsilon}}\right) = \tfrac{1-2\nu}{E}tr\left(\overline{\overline{\sigma}}\right) \quad and \quad \overline{\overline{\Delta}}\left(\overline{\overline{\varepsilon}}\right) = \frac{(1+\nu)}{E}\overline{\overline{\Delta}}\overline{\overline{\sigma}} - \frac{\nu}{E}\Delta tr\left(\overline{\overline{\sigma}}\right)\overline{\overline{I}}$$

Let $\Sigma = tr\left(\overline{\overline{\sigma}}\right)$. Then:

$$\begin{gathered}\overline{\overline{grad}}\left(\overrightarrow{div}\left((1+\mu)\overline{\overline{\sigma}} - \nu\Sigma\overline{\overline{I}}\right)\right) + \overline{\overline{grad}}^T\left(\overrightarrow{div}\left((1+\mu)\overline{\overline{\sigma}} - \nu\Sigma\overline{\overline{I}}\right)\right)\\ -\overline{\overline{grad}}\left(\overrightarrow{grad}\left(\left(1-2\nu\right)\Sigma\right)\right) - \left((1+\nu)\overline{\overline{\Delta}}\overline{\overline{\sigma}} - \nu\Delta\Sigma\overline{\overline{I}}\right) = \overline{\overline{0}}\end{gathered}$$

or again:

$$\begin{gathered}(1+\nu)\left(\overline{\overline{grad}}\,\overrightarrow{div}\,\overline{\overline{\sigma}} + \overline{\overline{grad}}^T\,\overrightarrow{div}\,\overline{\overline{\sigma}} - \overline{\overline{\Delta}}\overline{\overline{\sigma}}\right) - \nu\overbrace{\left[\overline{\overline{grad}}\left(\overrightarrow{grad}\Sigma\right) + \overline{\overline{grad}}^T\left(\overrightarrow{grad}\Sigma\right)\right]}^{2\overline{\overline{grad}}\left(\overrightarrow{grad}\Sigma\right)}\\ -(1-2\nu)\overline{\overline{grad}}\left(\overrightarrow{grad}\Sigma\right) + \nu\Delta\Sigma\overline{\overline{I}} = \overline{\overline{0}}\end{gathered}$$

therefore:

$$\boxed{\left(\overline{\overline{grad}}\,\overrightarrow{div}\,\overline{\overline{\sigma}} + \overline{\overline{grad}}^{T}\,\overrightarrow{div}\,\overline{\overline{\sigma}} - \overline{\overline{\Delta}}\,\overline{\overline{\sigma}}\right) - \frac{1}{(1+\nu)}\left(\overline{\overline{grad}}\left(\overrightarrow{grad}\Sigma\right) - \nu\Delta\Sigma\overline{\overline{I}}\right) = \overline{\overline{0}}} \tag{9.14}$$

Now take the divergence of equation 9.12:

$$\mu\, div\,\overrightarrow{\Delta}\,\vec{u} + (\mu+\lambda)\,\overbrace{div\overrightarrow{grad}}^{\Delta}\,div\,\vec{u} + div\,\vec{f_v} = 0$$

since $div\overrightarrow{grad}f = \Delta f$ and $\overrightarrow{\Delta}\,\vec{u} = \Delta div\,\vec{u}$, and since, moreover, $div\,\vec{u} = tr\left(\overline{\overline{\varepsilon}}\right) = \theta = \frac{1-2\nu}{E}\Sigma$, it follows that:

$$\boxed{(2\mu+\lambda)\,\frac{1-2\nu}{E}\Delta\Sigma + div\,\vec{f_v} \;=\; \vec{0}} \tag{9.15}$$

Finally, taking into account the relations, one obtains:

$$\frac{2\mu+\lambda}{2\mu+3\lambda}\Delta\Sigma + div\,\vec{f_v} \;=\; \vec{0}$$

$$\boxed{\frac{1-\nu}{1+\nu}\Delta\Sigma + div\,\vec{f_v} \;=\; \vec{0}} \tag{9.16}$$

Now taking local equilibrium equation 9.2 into account and substituting into equation 9.14, one obtains:

$$\boxed{\overline{\overline{\Delta}}\,\overline{\overline{\sigma}} + \frac{1}{1+\nu}\,\overline{\overline{grad}}\,\overrightarrow{grad}\Sigma + \overline{\overline{grad}}\,\vec{f_v} + \overline{\overline{grad}}^{T}\,\vec{f_v} + \frac{\nu}{1-\nu}div\,\vec{f_v}\overline{\overline{I}} = \overline{\overline{0}}} \tag{9.17}$$

These are the 6 BELTRAMI equations.

9.3.3 Case of uniform body forces

When the body forces are uniform (specific weight in the case of a homogeneous material), one has:

$$\overline{\overline{\Delta}}\,\overline{\overline{\sigma}} + \frac{1}{1+\nu}\,\overline{\overline{grad}}\,\overrightarrow{grad}\Sigma = \overline{\overline{0}}$$

If the Laplacian of this equation is taken:

$$\overline{\overline{\Delta\Delta}}\,\overline{\overline{\sigma}} + \frac{1}{1+\nu}\,\overline{\overline{grad}}\,\overrightarrow{grad}\Delta\Sigma = \overline{\overline{0}}$$

and since $div\,\vec{f_v} = 0$, there remains:

$$\overline{\overline{\Delta}}^{2}\,\overline{\overline{\sigma}} = \overline{\overline{0}}$$

⇒ the stress tensor is “biharmonic.”
Moreover, equation 9.16 becomes $\Delta\Sigma = 0$: the trace of the stress tensor is a harmonic function.

9.3.4 Statically admissible stress field

A “statically admissible stress field” (SA field) is any virtual stress field $\left\{\overline{\overline{\sigma^*}}\right\}$ that is continuous and differentiable (at least piecewise) and satisfies:

1. the local equilibrium equations:

$$\boxed{\overrightarrow{div}\,\overline{\overline{\sigma^*}} + \overrightarrow{f_v} = \overrightarrow{0}} \tag{9.18}$$

2. the boundary conditions of the solid:

$$\overrightarrow{T^*(P)} = \overline{\overline{\sigma^*(P)}}.\overrightarrow{n(P)} = \overrightarrow{f_s(P)} \quad \forall P \in \left\{\partial S_f\right\} \quad and \quad \overrightarrow{n} \perp \{\partial S\}$$

Such fields are used in approximate computational methods.

NOTE: The “real” stress field (exact solution of the BELTRAMI equations) is statically admissible.

9.4 Uniqueness of the Solution of an Elasticity Problem

9.4.1 Principle of superposition

Observe that Navier’s equations are linear partial differential equations, of second order and with constant coefficients. ⇒ If a loading $\{F_1\}$ applied to a solid (D) leads to a solution $\left\{\overline{\overline{\sigma_1}}, \overline{\overline{\varepsilon_1}}, \overrightarrow{u_1}\right\}$ and another loading $\{F_2\}$ applied to the same solid leads to a solution $\left\{\overline{\overline{\sigma_2}}, \overline{\overline{\varepsilon_2}}, \overrightarrow{u_2}\right\}$, then the loading $\{F_1 + F_2\}$ admits $\left\{\overline{\overline{\sigma_1}} + \overline{\overline{\sigma_2}}, \overline{\overline{\varepsilon_1}} + \overline{\overline{\varepsilon_2}}, \overrightarrow{u_1} + \overrightarrow{u_2}\right\}$ as a solution.

9.4.2 Uniqueness of the solution

Assume that the solid (D) is stress-free at rest (initial state), and let a loading $\{F\}$ be applied to it. Suppose that solving the problem, via the BELTRAMI or NAVIER equations, leads to two different solutions $\left\{\overline{\overline{\sigma_1}}, \overline{\overline{\varepsilon_1}}, \overrightarrow{u_1}\right\}$ and $\left\{\overline{\overline{\sigma_2}}, \overline{\overline{\varepsilon_2}}, \overrightarrow{u_2}\right\}$. The superposition principle implies that the difference of the two fields $\left\{\overline{\overline{\sigma_1}} - \overline{\overline{\sigma_2}}, \overline{\overline{\varepsilon_1}} - \overline{\overline{\varepsilon_2}}, \overrightarrow{u_1} - \overrightarrow{u_2}\right\}$ is a solution of the problem even though the solid (D) is subjected to no force.

The energy theorem states that, in the absence of applied forces, their work is zero; hence the elastic energy stored in the solid is also zero, and thus:

$$W_e = \int_S \frac{1}{2}\left[\overline{\overline{\sigma_1}} - \overline{\overline{\sigma_2}}\right] : \overline{\overline{\overline{\overline{S}}}}\left[\overline{\overline{\sigma_1}} - \overline{\overline{\sigma_2}}\right] dv = 0$$

Now, W_e is a positive-definite quadratic form and can vanish only if $\overline{\overline{\sigma_1}} - \overline{\overline{\sigma_2}} = \overline{\overline{0}}$. ⇒ Therefore the two solutions found are in fact one and the same unique solution.

9.4.3 Associated KA and SA fields

If, for an elasticity problem, one can associate through the constitutive law a kinematically admissible displacement field $\left\{\overrightarrow{u^*}\right\}$ with a statically admissible stress field $\left\{\overline{\overline{\sigma^*}}\right\}$, then the pair of fields $\left\{\overrightarrow{u^*}, \overline{\overline{\sigma^*}}\right\}$ constitutes the unique solution to the problem.

9.5 RITZ Method

The Ritz method is an approximate method for finding a solution to a continuum mechanics problem.

9.5.1 Theorem

In statics, one introduces the functional “Potential” (called potential or potential energy), denoted by γ, and defined as:

$$\boxed{\gamma(u^*) = V_e(u^*) + V_f(u^*)} \tag{9.19}$$

where:

- $\{u^*\}$ is a kinematically admissible displacement field (KA).
- V_e is the elastic potential associated with this KA displacement field:

$$V_e(u^*) = \int_S \frac{1}{2}\overline{\overline{\varepsilon^*}} : \overline{\overline{\overline{\overline{C}}}}\,\overline{\overline{\varepsilon^*}}.dv = \int_S \frac{1}{2}\overrightarrow{\varepsilon^*} : \underline{\underline{C}}\,\overrightarrow{\varepsilon^*}.dv$$

- V_f is the potential of the external forces:

$$V_f(u^*) = -\left(\underbrace{\int_S \overrightarrow{f_v}.\overrightarrow{u^*}.dv}_{body\ forces} + \underbrace{\int_{\partial S} \overrightarrow{f_s}.\overrightarrow{u^*}.dS}_{surface\ forces}\right)$$

Suppose that $\{u^*\}$ differs from the real displacement field by a kinematically admissible perturbation $\{\delta u\}$:

$$\boxed{\{u^*\} = \{u\} + \{\delta u\} \quad with \quad \{\delta u\} \quad \text{KA}} \tag{9.20}$$

The strain tensor derived from $\{u^*\}$ can then be established. By linearity:

$$\boxed{\overline{\overline{\varepsilon^*}} = \underbrace{\overline{\overline{\varepsilon}}}_{real\ strain\ field} + \underbrace{\delta\overline{\overline{\varepsilon}}}_{perturbation\ strain\ field}} \tag{9.21}$$

Then, by symmetry:

$$\boxed{V_e(u^*) = \int_S \frac{1}{2}\left(\delta\overline{\overline{\varepsilon}} : \overline{\overline{\overline{\overline{C}}}}\delta\overline{\overline{\varepsilon}} + 2\delta\overline{\overline{\varepsilon}} : \underbrace{\overline{\overline{\overline{\overline{C}}}}\,\overline{\overline{\varepsilon}}}_{\overline{\overline{\sigma}}} + \overline{\overline{\varepsilon}} : \overline{\overline{\overline{\overline{C}}}}\,\overline{\overline{\varepsilon}}\right) dv} \tag{9.22}$$

Moreover:

$$\boxed{V_f(u^*) = -\left(\int_S \overrightarrow{f_v}.\overrightarrow{\delta u}.dv + \int_{\partial S} \overrightarrow{f_s}.\overrightarrow{\delta u}.dS\right) - \left(\int_S \overrightarrow{f_v}.\overrightarrow{u}.dv + \int_{\partial S} \overrightarrow{f_s}.\overrightarrow{u}.dS\right)} \tag{9.23}$$

By summation and regrouping:

$$\gamma(u^*) = \underbrace{\left[\int_S \frac{1}{2}\overline{\overline{\varepsilon}} : \overline{\overline{\overline{\overline{C}}}}\,\overline{\overline{\varepsilon}} - \left(\int_S \overrightarrow{f_v}.\overrightarrow{u}.dv + \int_{\partial S} \overrightarrow{f_s}.\overrightarrow{u}.dS\right)\right]}_{\gamma(u)}$$

$$-\underbrace{\left[\overbrace{-\int_S \overline{\overline{\sigma}} : \delta\overline{\overline{\varepsilon}}.dv}^{P_{i^*}} + \overbrace{\left(\int_S \overrightarrow{f_v}.\overrightarrow{\delta u}.dv + \int_{\partial S} \overrightarrow{f_s}.\overrightarrow{\delta u}.dS\right)}^{P_{e^*}}\right]}_{P_{i^*}+P_{e^*}=0} + \underbrace{\int_S \frac{1}{2}\delta\overline{\overline{\varepsilon}} : \overline{\overline{\overline{\overline{C}}}}\,\overline{\overline{\varepsilon}}}_{symmetric\ bilinear\ form\ \geq 0} \tag{9.24}$$

- The first term is the potential $\gamma(u)$ computed for the real displacement field.
- The second term vanishes by virtue of the principle of virtual powers.
- The third term is a bilinear form of the perturbation strain field (and therefore of the displacement field, under the small-perturbation hypothesis). This bilinear form is definite, symmetric, and positive.

Consequently:

$$\boxed{\gamma(u^*) \geq \gamma(u) \quad \forall u^* \quad \text{KA}} \tag{9.25}$$

RITZ statement: Among all kinematically admissible displacement fields, the "real" displacement field (solution of the elasticity problem) minimizes the potential of the solid system.

Statement 2: Within a restricted set (a "family") of kinematically admissible displacement fields, the "best" displacement field is the one that minimizes the potential.

Reminder: A clear distinction must be made between "elastic potential" and "elastic energy," on the one hand, and between "work of external forces" and "potential of external forces," on the other hand. Ritz's theorem is expressed in terms of potentials.

Consider the example of a spring of stiffness k subjected to a force F, and suppose that its elongation u is sought.

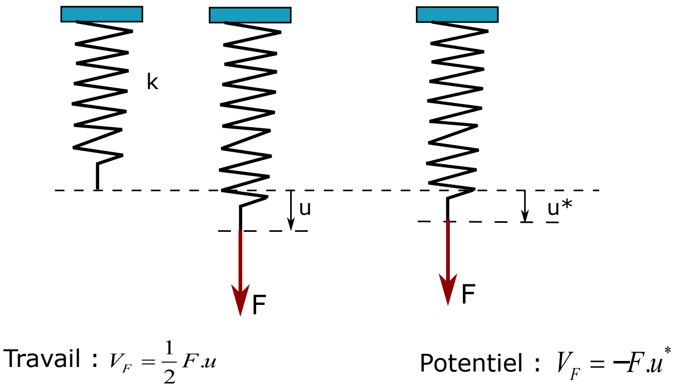


Figure 9.3: Work versus potential of a force – spring case.

Mechanical energy theorem:
Elastic energy of the spring: $W_e = \frac{1}{2}ku^2$
Work of the force: $T_F = \frac{1}{2}F.u$
Identification: $W_e = T_F \Rightarrow u = \frac{F}{k}$

Ritz method:
Elastic potential of the spring: $V_e = \frac{1}{2}ku^{*2}$
Potential of the force: $V_F = -F.u^*$
Potential of the system: $\gamma = \frac{1}{2}ku^{*2} - F.u^*$
Potential minimization: $\left.\frac{d\gamma}{du^*}\right|_{u^*=u_{reel}} = 0 \Rightarrow ku - F = 0 \Rightarrow u = \frac{F}{k}$

9.5.2 Application

This theorem forms the basis of a method for seeking approximate solutions to elasticity problems. It follows a displacement-based approach, the displacement field being sought as a candidate within a family of kinematically admissible displacement fields.

It is thus common to seek the displacement field as a linear combination[1] of independent kinematically admissible functions chosen from a given class of functions (for example polynomials or harmonic functions):

$$\boxed{\overrightarrow{u^*}(M) = \sum_{i=1}^{n} Z_i . \overrightarrow{\varphi_i}(x,y,z) \quad \text{with} \quad \varphi_i \quad \text{KA}} \tag{9.26}$$

The scalar parameters Z_i are amplitudes (proportional to displacements), whereas the functions $\overrightarrow{\varphi_i}$ are judiciously chosen "shape functions." By replacing the function $\overrightarrow{u^*}$ with a linear combination of known functions, the functional $\gamma\left(\overrightarrow{u^*}\right)$ is replaced, mathematically, by a function of several variables $\tilde{\gamma}(Z_i)$.

The practical minimization problem of the functional γ is thus reduced to the search for the zeros of the partial derivatives of the function $\tilde{\gamma}$. The best solution (given the a priori choice of shape functions) is therefore obtained by solving the set of equations:

$$\boxed{\frac{d\tilde{\gamma}}{dZ_i} \quad \forall j} \tag{9.27}$$

In linear elasticity, under the small-perturbation assumption, these equations are linear and form a definite, positive, symmetric system[2]. One obtains:

$$\begin{pmatrix} K_{11} & K_{12} & \cdots & \\ K_{12} & K_{22} & & \\ \vdots & & \ddots & K_{n,n-1} \\ & & K_{n,n-1} & K_{n,n} \end{pmatrix} . \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} \tag{9.28}$$

This system $[K][Z] = [F]$ consists of a "stiffness matrix" $[K]$, a "displacement vector" $[Z]$ whose components are the amplitudes Z_i, also called "degrees of freedom," and finally the vector of "generalized forces" $[F]$, which depends only on the loading applied to the structure and on the choice of shape functions.
From there:

- the unknown vector $[Z_i]$ is computed by inverting the linear system: $[Z] = [K]^{-1}[F]$
- the approximate displacement field is then obtained: $\overrightarrow{u^*}(M) = \sum_{i=1}^{n} Z_i . \overrightarrow{\varphi_i}(M)$
- from this, the approximate strain field is derived: $\overline{\overline{\varepsilon^*}} = \frac{1}{2}\left[\overline{\overline{grad}}\,\overrightarrow{u^*} + \overline{\overline{grad}}^{T}\,\overrightarrow{u^*}\right]$
- and finally the stresses: $\overline{\overline{\sigma^*}} = 2\mu\overline{\overline{\varepsilon^*}} + \lambda\theta^*\overline{\overline{I}}$
- if the resulting stress field is statically admissible and satisfies the boundary conditions, then the set $\left\{\overrightarrow{u^*}, \overline{\overline{\sigma^*}}, \overline{\overline{\varepsilon^*}}\right\}$ constitutes the "real" solution of the elasticity problem; otherwise, it is only an approximation.

9.5.3 Finite element method

The finite element method consists in decomposing the solid under study into small elements of prescribed shape (in $3D$: tetrahedra, parallelepipeds; in $2D$: triangles, rectangles, quadrilaterals,

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[1] Naturally, other types of combinations are also possible.

[2] Under large displacements, the system may cease to be definite for certain loading configurations leading to shape instability.

etc.), assuming that the displacement patterns are known on each of them.

On each element, the displacement field is described by the amplitudes of the displacements at particular points called "nodes" (for example, but not only, the vertices). Each element therefore possesses its own stiffness matrix (element matrix).

Continuity of the solid and of the displacement field is achieved by sharing the nodes of adjacent elements. The displacement amplitudes of all nodes constitute the "degrees of freedom" of the system. Formally, this sharing[3] leads to "matrix assembly," from which the "global stiffness matrix" of the solid and the vector of generalized forces are obtained.
This method of discretizing a solid into small elements is well suited to automatic computer-based calculation. Computational programs thus allow one to model the problem, generate the mesh, perform matrix assembly, solve the linear system, and exploit the results in the form of graphs, tables, and various animations.

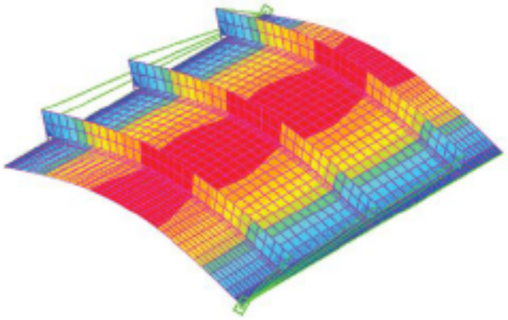

Figure 9.4: Mesh of a shell.

This method has important developments beyond elasticity and even beyond the strict framework of mechanics. There are thus codes capable of treating problems in heat transfer (conduction), diffusion, fluid mechanics, electromagnetism, and even coupled phenomena.

9.6 Two-Dimensional Elasticity

Two-dimensional elasticity concerns solids of particular geometry subjected to particular loadings; it can then be shown that the stress tensor (or the strain tensor) contains only four components depending on two spatial variables.

In such a configuration, not only is the problem greatly simplified mathematically, but it also benefits from specific solution methods.

There are two cases of so-called plane elasticity:

1. the plane strain state concerns prismatic bodies whose length (along the prism axis z) is infinitely larger than the transverse dimensions of the cross-section. This prism is subjected only to lateral forces acting perpendicularly to the prism axis and uniformly along its entire length. In this case, the axial displacement u_z is assumed to be zero, and there is neither axial strain along this axis nor distortion of the right angles built on this axis and any orthogonal

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[3] Support conditions are also taken into account.

direction. Thus, in Cartesian coordinates, $\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$. In this case, it is the strain tensor that has only three nonzero components $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$. Note that, owing to Poisson's effect, the axial normal stress is not zero, but is given by $\sigma_{zz} = +\nu.(\sigma_{xx} + \sigma_{yy})$

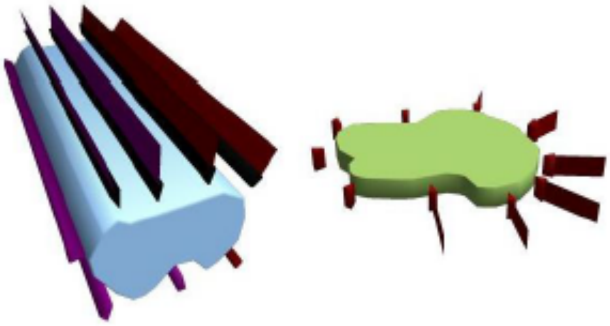

Figure 9.5: Plane strain and plane stress states.

2. the plane stress state concerns thin plates loaded in their own plane (thus excluding bending, as would occur in a floor slab, for example). The parallel surfaces delimiting the plate are not loaded. Let $\overrightarrow{z}$ denote the axis perpendicular to the plate; the quantities are assumed not to depend on the coordinate z. Moreover, the stress vector $\overrightarrow{\Phi_Z} = \overline{\overline{\sigma}}.\overrightarrow{z}$ is uniformly zero. The stress tensor therefore has only three nonzero components, for example $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ in Cartesian coordinates.
   Note that, owing to Poisson's effect, the transverse strain ε_{zz} is not zero, but is locally given by $\varepsilon_{zz} = -\nu.(\varepsilon_{xx} + \varepsilon_{yy})$.

9.6.1 Form of the tensors – reduced tensors

Although each tensor contains only three independent components, the form of the tensors changes depending on the type of plane elasticity.

Plane stress state

In Cartesian coordinates, the stress tensor takes the form:

$$\overline{\overline{\sigma}}(M) = \begin{pmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) & 0 \\ \sigma_{12}(x_1,x_2) & \sigma_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\overline{\overline{\tilde{\sigma}}}$ denote the in-plane part of the stress tensor.

$$\overline{\overline{\tilde{\sigma}}} = \begin{pmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) \\ \sigma_{12}(x_1,x_2) & \sigma_{22}(x_1,x_2) \end{pmatrix}$$

Owing to Hooke's law, the strain tensor (not purely planar) is:

$$\overline{\overline{\varepsilon}}(M) = \begin{pmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{12}(x_1,x_2) & 0 \\ \varepsilon_{12}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) & 0 \\ 0 & 0 & \varepsilon_{33}(x_1,x_2) \end{pmatrix}$$

Owing to Poisson's ratio, the axial strain along axis $\overrightarrow{e_3}$ is not zero, but is given by:

$$\varepsilon_{33} = -\nu.(\varepsilon_{11} + \varepsilon_{22})$$

Let $\overline{\overline{\tilde{\varepsilon}}}$ denote the in-plane part of the strain tensor.

$$\overline{\overline{\tilde{\varepsilon}}}(M) = \begin{pmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{12}(x_1,x_2) \\ \varepsilon_{12}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) \end{pmatrix}$$

Plane strain state

In Cartesian coordinates, the strain tensor takes the form:

$$\overline{\overline{\varepsilon}}(M) = \begin{pmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{12}(x_1,x_2) & 0 \\ \varepsilon_{12}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\overline{\overline{\tilde{\varepsilon}}}$ denote the in-plane part of the strain tensor.

$$\overline{\overline{\tilde{\varepsilon}}}(M) = \begin{pmatrix} \varepsilon_{11}(x_1,x_2) & \varepsilon_{12}(x_1,x_2) \\ \varepsilon_{12}(x_1,x_2) & \varepsilon_{22}(x_1,x_2) \end{pmatrix}$$

Owing to Hooke's law, the stress tensor (not purely planar) is:

$$\overline{\overline{\sigma}}(M) = \begin{pmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) & 0 \\ \sigma_{12}(x_1,x_2) & \sigma_{22}(x_1,x_2) & 0 \\ 0 & 0 & \sigma_{33}(x_1,x_2) \end{pmatrix}$$

Owing to Poisson's ratio, the normal stress along axis $\overrightarrow{e_3}$ is not zero, but is given by:

$$\sigma_{33} = +\nu.(\sigma_{11} + \sigma_{22})$$

Let $\overline{\overline{\tilde{\sigma}}}$ denote the in-plane part of the stress tensor:

$$\overline{\overline{\tilde{\sigma}}} = \begin{pmatrix} \sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) \\ \sigma_{12}(x_1,x_2) & \sigma_{22}(x_1,x_2) \end{pmatrix}$$

9.6.2 Compatibility equation

Recall the compatibility equations established in $3D$ and in Cartesian coordinates:

$$\begin{cases} \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \\ \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \\ \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} = 0 \end{cases} \qquad \begin{cases} \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} + \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0 \\ \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} + \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{23}}{\partial x_1}\right) = 0 \\ \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2}\right) = 0 \end{cases}$$

Given that $\varepsilon_{31} = \varepsilon_{32} = 0$ and $\frac{\partial}{\partial x_3} \equiv 0$, in plane elasticity one obtains:

$$\begin{cases} \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \\ \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 0 \\ \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 0 \\ \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 \end{cases}$$

- In plane strain, the term ε_{33} is also zero, so only one equation remains:

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0} \tag{9.29}$$

- In plane stress, the function $\varepsilon_{33}(x_1, x_2)$ has zero second derivatives; it is therefore a linear function of its variables: $\varepsilon_{33}(x_1, x_2) = ax_1 + bx_2 + c$. Since this condition is difficult to satisfy, plane stress states constitute an approximation of the actual stress state occurring in plates loaded in their own plane.

9.6.3 Trace of the tensors

If needed, note that $tr\left(\overline{\overline{\tilde{I}}}\right) = 2$.

Plane stress state

$$\Sigma = \sigma_{11} + \sigma_{22} = \tilde{\Sigma}$$

$$\theta = (1-\nu)(\varepsilon_{11} + \varepsilon_{22}) = (1-\nu)\,\tilde{\theta}$$

Plane strain state

$$\Sigma = (1+\nu)(\sigma_{11} + \sigma_{22}) = (1+\nu)\,\tilde{\Sigma}$$

$$\theta = \varepsilon_{11} + \varepsilon_{22} = \tilde{\theta}$$

9.6.4 Equilibrium

Plane stress state

The equilibrium equations reduce to:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1^v = 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2^v = 0 \\ f_3^v = 0 \end{cases}$$

$$\Rightarrow div\overline{\overline{\tilde{\sigma}}} + \overrightarrow{f_v} = \overrightarrow{0}$$

On the boundary of the solid (lateral surface (∂S)):

$$\overrightarrow{T\left(P, \overrightarrow{n}\right)} = \overline{\overline{\sigma}}(P) . \overrightarrow{n} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} T_1 \\ T_2 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{1s} \\ f_{2s} \\ f_{3s} \end{pmatrix}$$

This confirms that no force can act along $\overrightarrow{e_3}$

Plane strain state

The equilibrium equations reduce to:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1^v = 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2^v = 0 \\ f_3^v = 0 \end{cases}$$

$$\Rightarrow div\overline{\overline{\tilde{\sigma}}} + \overrightarrow{f_v} = \overrightarrow{0}$$

On the boundary of the solid (lateral surface (∂S)):

$$\overrightarrow{T\left(P, \overrightarrow{n}\right)} = \overline{\overline{\sigma}}(P) . \overrightarrow{n} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} . \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} T_1 \\ T_2 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{1s} \\ f_{2s} \\ f_{3s} \end{pmatrix}$$

This confirms that no axial force can exist.

9.6.5 Stress–strain relation

Plane stress state

The trace of the stress tensor is $\Sigma = \sigma_{11} + \sigma_{22}$. Applying Hooke's law:

$$\begin{cases} \varepsilon_{11} = \frac{\sigma_{11} - \nu\sigma_{22}}{E} \\ \varepsilon_{22} = \frac{\sigma_{22} - \nu\sigma_{11}}{E} \\ \varepsilon_{12} = \frac{(1+\nu)}{E}\sigma_{12} \end{cases}$$

Plane strain state

The trace of the stress tensor is $\Sigma = (1+\nu)(\sigma_{11} + \sigma_{22})$; applying Hooke's law:

$$\Sigma = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(1+\nu)(\sigma_{11} + \sigma_{22})$$

$$\Rightarrow \left[\frac{1-\nu^2}{E}\right]\left[\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right]$$

Plane stress state

Let:

$$\boxed{E = E'\frac{1+2\nu'}{(1+\nu')^2} \quad \text{and} \quad \nu = \frac{\nu'}{1+\nu'}}$$

“reduced” constitutive law:

$$\bar{\bar{\tilde{\sigma}}} = \frac{E}{1+\nu}\left[\bar{\bar{\tilde{\varepsilon}}} + \frac{\nu}{1-\nu}\tilde{\theta}\bar{\bar{\tilde{I}}}\right]$$

or:

$$\bar{\bar{\tilde{\varepsilon}}} = \frac{1+\nu}{E}\bar{\bar{\tilde{\sigma}}} - \frac{\nu}{E}\tilde{\Sigma}\bar{\bar{\tilde{I}}}$$

VOIGT notation:

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix}\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{pmatrix}\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

Plane strain state

likewise:

$$\varepsilon_{22} = \left[\frac{1-\nu^2}{E}\right]\left[\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right]$$

and finally:

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \left[\frac{1-\nu^2}{E}\right]\left[\frac{1}{1-\nu}\right]\sigma_{12}$$

$$\Rightarrow \varepsilon_{12} = \left[\frac{1-\nu^2}{E}\right]\left[1 + \frac{1}{1-\nu}\right]\sigma_{12}$$

Let:

$$\boxed{E' = \frac{E}{1-\nu^2} \quad \text{and} \quad \nu' = \frac{\nu}{1-\nu}}$$

then, in plane strain:

$$\begin{cases} \varepsilon_{11} = \frac{\sigma_{11} - \nu'\sigma_{22}}{E'} \\ \varepsilon_{22} = \frac{\sigma_{22} - \nu'\sigma_{11}}{E'} \\ \varepsilon_{12} = \frac{(1+\nu')}{E'}\sigma_{12} \end{cases}$$

“reduced” constitutive law:[4]

$$\bar{\bar{\tilde{\varepsilon}}} = \frac{1+\nu}{E}\left[\bar{\bar{\tilde{\sigma}}} - \nu\tilde{\Sigma}\bar{\bar{\tilde{I}}}\right]$$

or:

$$\bar{\bar{\tilde{\varepsilon}}} = \frac{1+\nu'}{E'}\bar{\bar{\tilde{\sigma}}} - \frac{\nu'}{E'}\tilde{\Sigma}\bar{\bar{\tilde{I}}}$$

VOIGT notation:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \frac{E}{1+\nu}\begin{pmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

9.6.6 Volumetric energy

In both cases:

$$\boxed{W_e = \frac{1}{2}(\sigma_{11}.\varepsilon_{11} + \sigma_{22}.\varepsilon_{22} + \sigma_{12}.\gamma_{12})} \tag{9.30}$$

9.6.7 BELTRAMI equations

We shall derive Beltrami’s equations in the plane stress state. The transposition to the plane strain state is obtained simply by replacing E with $E' = E/1-\nu^2$ and ν with $\nu' = \nu/1-\nu$

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[4] Note that $\bar{\bar{\tilde{I}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Plane stress state

Starting from the compatibility equation in $2D$ and in Cartesian coordinates (9.29), and replacing strains by stresses using the constitutive law:

$$\frac{1}{E}\frac{\partial^2}{\partial x_2^2}[\sigma_{11}-\nu\sigma_{22}]+\frac{1}{E}\frac{\partial^2}{\partial x_1^2}[\sigma_{22}-\nu\sigma_{11}]-2\frac{1+\nu}{E}\frac{\partial^2}{\partial x_1}\frac{\sigma_{12}}{\partial x_2}=0$$

that is:

$$\Delta(\sigma_{11}+\sigma_{22})-(1+\nu)\left[\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}+2\frac{\partial^2\sigma_{12}}{\partial x_1\partial x_2}\right]=0$$

now, $\overrightarrow{\mathrm{div}}\tilde{\sigma}+\overrightarrow{\mathrm{f_v}}=\overrightarrow{0}$, i.e.:

$$\begin{cases}\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+f_1^v=0 & \mathrm{x}\frac{\partial}{\partial x_1}\\ \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+f_2^v=0 & \mathrm{x}\frac{\partial}{\partial x_2}\end{cases}$$

by summing:

$$\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}+\frac{\partial^2\sigma_{22}}{\partial x_1\partial x_2}+div\,\overrightarrow{f_v}=0$$

finally:

$$\Delta\tilde{\Sigma}+(1+\nu)\,div\,\overrightarrow{f_v}=0 \qquad (9.31)$$

Plane strain state

transposing to plane strain:

$$\Delta\tilde{\Sigma}+\left(1+\nu'\right)div\,\overrightarrow{f_v}=0$$

that is:

$$\Delta\tilde{\Sigma}+\frac{1}{1-\nu}div\,\overrightarrow{f_v}=0 \qquad (9.32)$$

9.6.8 AIRY function

Stresses derive from a single function

The objective is to identify a single function from which the stress field derives. To this end, let us return to the equilibrium equations, which are identical in both plane elasticity states.

$$div\,\overline{\overline{\overline{\sigma}}}+\overrightarrow{f_v}=\overrightarrow{0}$$

Assume, which is in fact a very common situation, that the body forces derive from a potential[5]:

$$\overrightarrow{f_v}=\overrightarrow{gradV}\Leftrightarrow f_1^v={}^{\partial V}\!/_{\partial x_1}\quad\text{and}\quad f_2^v={}^{\partial V}\!/_{\partial x_2}$$

The equations then become:

$$\begin{cases}\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial V}{\partial x_1}=0\\ \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial V}{\partial x_2}=0\end{cases}\Rightarrow\begin{cases}\frac{\partial(\sigma_{11}+V)}{\partial x_1}=-\frac{\partial\sigma_{12}}{\partial x_2}\\ \frac{\partial(\sigma_{22}+V)}{\partial x_2}=-\frac{\partial\sigma_{12}}{\partial x_1}\end{cases}$$

Recalling Cauchy's theorem[6], and introducing two functions φ and ψ, one may set:

$$\begin{cases}\sigma_{11}+V=\frac{\partial\varphi}{\partial x_2} & \sigma_{22}+V=\frac{\partial\psi}{\partial x_1}\\ \sigma_{12}=-\frac{\partial\varphi}{\partial x_1} & \sigma_{12}=-\frac{\partial\psi}{\partial x_2}\end{cases}$$

Since, obviously, $\frac{\partial\varphi}{\partial x_1}=\frac{\partial\psi}{\partial x_2}$, one may introduce a new function Φ such that:

$$\varphi=\frac{\partial\Phi}{\partial x_2}\quad\text{and}\quad\psi=\frac{\partial\Phi}{\partial x_1}$$

---

[5] For example, if gravity acts along axis $\overrightarrow{e_1}$, then: $\overrightarrow{f_v}=\rho g\overrightarrow{e_1}$, and the corresponding potential of the specific weight is $V=\rho g x_1$.

[6] $\frac{\partial^2 h}{\partial x_1\partial x_2}=\frac{\partial^2 h}{\partial x_2\partial x_1}$ (in a double differentiation, the order of differentiation is immaterial if the function h is sufficiently smooth).

Consequently:

$$\boxed{\begin{cases} \sigma_{11}+V=\frac{\partial\varphi}{\partial x_2}=\frac{\partial^2\Phi}{\partial x_2^2} \\ \sigma_{22}+V=\frac{\partial\psi}{\partial x_1}=\frac{\partial^2\Phi}{\partial x_1^2} \\ \sigma_{12}=-\frac{\partial^2\Phi}{\partial x_1\partial x_2} \end{cases}} \tag{9.33}$$

Property of the AIRY function

- case where body forces derive from a potential

Plane stress state

BELTRAMI equation:

$$\Delta\tilde{\Sigma}+(1+\nu)\,\mathrm{div}\overrightarrow{f_v}=0$$

with:

$$\tilde{\Sigma}=\sigma_{11}+\sigma_{22}=\Delta\Phi-2V$$

thus[7]:

$$\Delta(\Delta\Phi-2V)+(1+\nu)\Delta V=0$$

hence:

$$\boxed{\Delta^2\Phi+(1-\nu)\Delta V=0} \tag{9.34}$$

Plane strain state

BELTRAMI equation:

$$\Delta\tilde{\Sigma}+\left(1+\nu'\right)\mathrm{div}\overrightarrow{f_v}=0$$

transposed to plane strain:

$$\Delta^2\Phi+\left(1-\nu'\right)\Delta V=0$$

that is:

$$\boxed{\Delta^2\Phi+\frac{1-2\nu}{1-\nu}\Delta V=0} \tag{9.35}$$

- case where body forces are harmonic, uniform, or zero

Then: $\Delta V=0$. In both cases of plane elasticity, there remains:

$$\boxed{\Delta^2\Phi=0} \tag{9.36}$$

⇒ The AIRY function is biharmonic.

Formulation of a plane elasticity problem

Seek a function $\Phi(x_1,x_2)$ such that, if body forces are harmonic, $\Delta^2\Phi=0$, and from which the stresses are derived through:

$$\begin{cases} \sigma_{11}=\frac{\partial^2\Phi}{\partial x_2^2}-V \\ \sigma_{22}=\frac{\partial^2\Phi}{\partial x_1^2}-V \\ \sigma_{12}=-\frac{\partial^2\Phi}{\partial x_1\partial x_2}-V \end{cases} \tag{9.37}$$

The stress field then automatically satisfies the equilibrium equations and the compatibility equations. It must also satisfy the stress boundary conditions: $\overrightarrow{T\left(P,\overrightarrow{n}\right)}=\overrightarrow{f_s}$ on ∂S_f. The displacements derived from the stresses must in turn satisfy the displacement conditions $\overrightarrow{u}=\overrightarrow{u_0}$ on ∂S_f.

⇒ In practice, the search for such a function Φ is as difficult as the direct search for the stress field (a fourth-order partial differential equation); therefore, an approximate solution is often sought in the form of a polynomial

$$\Phi=\sum\alpha_i x_1^n x_2^n \tag{9.38}$$

whose order is at most 4, with coefficients judiciously chosen to exploit possible symmetries and the traction boundary conditions. This leads to interesting applications, notably in beam theory, for highlighting second-order effects relative to the classical NAVIER–BERNOULLI assumptions. The solution obtained in this way is statically admissible, but often not kinematically admissible.

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[7] $\mathrm{div}\overrightarrow{grad}h=\Delta h$

9.7 Axisymmetric Elasticity

An elasticity problem is said to be axisymmetric when two conditions are simultaneously satisfied:

1. the shape of the solid under study has rotational symmetry about an axis (z);
2. the loading and the boundary conditions are also rotationally symmetric about the same axis.

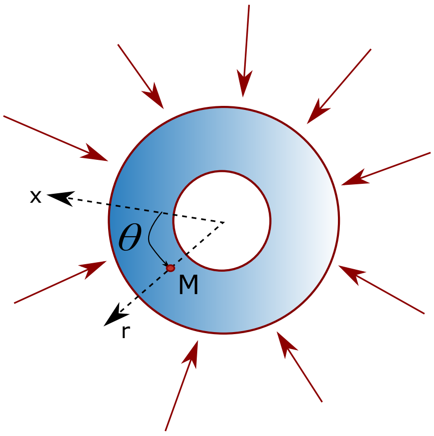

Figure 9.6: Example of an axisymmetric problem.

In this case, the solution is also axisymmetric. It is then advantageous to use a cylindrical coordinate system around the axis of symmetry. All quantities are therefore independent of the polar angle θ. The problem is thus reduced to two variables, r and z.
Axisymmetry can also be combined with a plane elasticity state, in which case the quantities depend only on r. This is the very special case of the infinitely long tube or the annular plate loaded radially.

9.8 Exercise

9.8.1 Statement

Consider the long rectangular block shown in the following figure: Assume the following hypotheses:

- Linear elastic behavior, with parameters E and ν
- The upper face is subjected to a uniform pressure p
- The lower face is on frictionless sliding support, fixed along the axis $(O, \overrightarrow{e_3})$
- Body forces are negligible.
- Small-perturbation hypothesis.
- We consider the central zone of the block in order to avoid edge effects.
- The displacement field is given by $\overrightarrow{U} = (A.X_1 + C.X_2)\,\overrightarrow{e_1} + (-C.X_1 + B.X_2)\,\overrightarrow{e_2}$

1. Represent the cross-section of the block containing point M, together with the boundary conditions on this section.
2. Compute the strain tensor $\overline{\overline{\varepsilon}}$, and give its principal values and principal directions. What type of deformation is it?
3. Express analytically the Cauchy stress tensor $\overline{\overline{\sigma}}$ (use Lamé's coefficients to simplify the notation).
4. Using the stress boundary conditions, determine the expressions of the parameters A and B. Using the displacement boundary conditions, determine C.
5. Let $E = 210000$ MPa and $\nu = 0.3$ (steel block). The section dimensions are $L = 0.2\,m$ and $H = 0.15\,m$, and the loading is $p = 500$ MPa. Compute the displacement vector of point M.

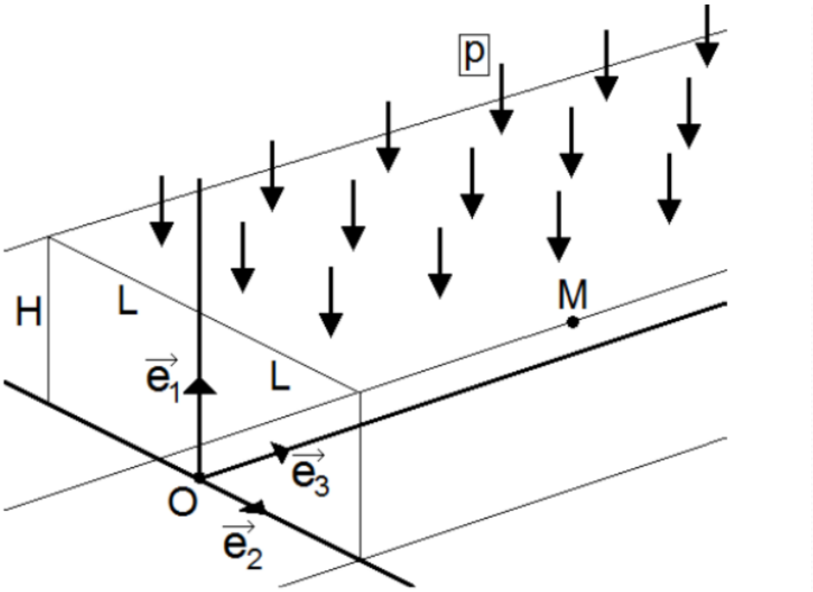

Figure 9.7: Elasticity problem.

Plot the deformed shape of the section.

9.8.2 Solution

- The section of the block containing point M is a rectangle. It is subjected to the following boundary conditions:
  - Upper face: uniform pressure equal to p.
  - Lateral faces: uniform pressure equal to 0.
  - Lower face: vertical displacement blocked, horizontal displacement blocked at the central point.

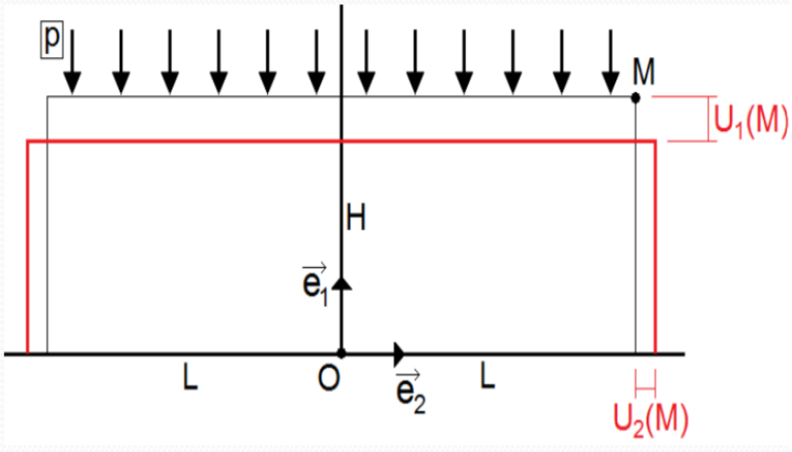

Figure 9.8: Section of the block.

- The displacement field is given by:

$$\vec{U} = \begin{cases} A.X_1 + C.X_2 \\ -C.X_1 + B.X_2 \\ 0 \end{cases}$$

This is clearly a plane motion. The linearized strain tensor is computed as:

$$\overline{\overline{\varepsilon}} = \frac{1}{2}\left(\overline{\overline{grad}}\overrightarrow{U} + \left(\overline{\overline{grad}}\overrightarrow{U}\right)^T\right)$$

Now:

$$\overline{\overline{grad}}\overrightarrow{U} = \begin{bmatrix} \frac{\partial U_1}{\partial X_1} & \frac{\partial U_1}{\partial X_2} & \frac{\partial U_1}{\partial X_3} \\ \frac{\partial U_2}{\partial X_1} & \frac{\partial U_2}{\partial X_2} & \frac{\partial U_2}{\partial X_3} \\ \frac{\partial U_3}{\partial X_1} & \frac{\partial U_3}{\partial X_2} & \frac{\partial U_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} A & C & 0 \\ -C & B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally:

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that this is a plane deformation, that the principal strain basis is the original basis $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$, and that the principal strains are $\{A, B, 0\}$.

- We apply Hooke's law written in terms of Lamé's coefficients:

$$\overline{\overline{\sigma}} = 2\mu.\overline{\overline{\varepsilon}} + \lambda.tr\left(\overline{\overline{\varepsilon}}\right).\overline{\overline{I}}$$

with:

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Since $tr\left(\overline{\overline{\varepsilon}}\right) = A + B$, one gets:

$$\overline{\overline{\sigma}} = 2\mu.\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda.(A+B).\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and finally:

$$\overline{\overline{\sigma}} = \begin{bmatrix} 2\mu A + \lambda.(A+B) & 0 & 0 \\ 0 & 2\mu B + \lambda.(A+B) & 0 \\ 0 & 0 & \lambda.(A+B) \end{bmatrix}$$

The tensors $\overline{\overline{\sigma}}$ and $\overline{\overline{\varepsilon}}$ are homogeneous, that is, they are identical at every point of the system.

- On the upper face, the applied pressure may be written as:

$$\overrightarrow{T_d} = -p.\overrightarrow{e_1}$$

Moreover, the traction boundary condition on this upper face is:

$$\overline{\overline{\sigma}}\overrightarrow{n} = \overrightarrow{T_d}$$

The unit normal to this surface is $-\overrightarrow{e_1}$, thus $\overline{\overline{\sigma}}(-\overrightarrow{e_1}) = \overrightarrow{T_d}$.
Therefore:

$$-\left(2\mu A + \lambda.(A+B)\right).\overrightarrow{e_1} = -p.\overrightarrow{e_1}$$

and finally:

$$2\mu A + \lambda.(A+B) = p$$

Applying the same reasoning to one of the lateral faces, one obtains:

$$2\mu B + \lambda.(A+B) = 0$$

This leads to a linear system of two equations in the two unknowns A and B:

$$\begin{cases} (2\mu+\lambda)A + \lambda B = p \\ \lambda A + (2\mu+\lambda)B = 0 \end{cases}$$

The second equation gives $B = \frac{-\lambda A}{2\mu+\lambda}$, which is substituted into the first to obtain:

$$A\left(2\mu+\lambda-\frac{\lambda^2}{2\mu+\lambda}\right)=p$$

After calculation, one obtains:

$$A = p\tfrac{-2\mu-\lambda}{4\mu(\mu+\lambda)} \quad \text{and} \quad B = \tfrac{\lambda}{4\mu(\mu+\lambda)}$$

On the lower face, the vertical displacement is blocked. Setting $X_1 = 0$, one may write:

$$U_1\left(0,X_2,X_3\right)=0$$

Now:

$$U_1\left(X_1,X_2,X_3\right)=A.X_1+C.X_2$$

Hence $C.X_2 = 0 \; \forall \left(X_2,X_3\right)$, and therefore $C = 0$.

- Using the previous results, the displacement field may be rewritten as:

$$\overrightarrow{U}=\begin{cases} A.X_1 = p\frac{-2\mu-\lambda}{4\mu(\mu+\lambda)}.X_1 \\ B.X_2 = \frac{\lambda}{4\mu(\mu+\lambda)}.X_2 \end{cases}$$

The coordinates of point M are $X_1 = H = 0.15\,m$ and $X_2 = L = 0.2\,m$.
Moreover, Lamé's coefficients are:

$$\mu=\frac{E}{2\left(1+\nu\right)}=\frac{210000}{2\left(1+0.3\right)}=80770MPa$$

$$\lambda=\frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}=\frac{210000*0.3}{\left(1+0.3\right)\left(1-2*0.3\right)}=121150MPa$$

Thus, finally, $A = -2.1666.10^{-3}$ and $B = 9.2855.10^{-4}$, and hence:

$$\overrightarrow{U}\left(M\right)=\begin{cases} -0.325mm \\ 0.186mm \\ 0 \end{cases}$$

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